Output-Feedback Control of an Extended Class of Sandwiched Hyperbolic PDE-ODE Systems

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***Abstract*—Motivated by an engineering application in brake control of cable mining elevators, where the dynamics consist of a brake, a shock absorber, a time-varying-length cable, and a cage, we address a theoretical problem of control of a particular class of coupled hyperbolic PDEs sandwiched between a nonlinear ODE on the actuated side and a linear ODE on the opposite side, with a PDE domain that is time-varying. A state-feedback controller entering a single ODE state is designed to exponentially stabilize the overall system through several backstepping transformations. An observer which only uses the boundary values at the actuated side is constructed to recover all the states of the overall system, based on which a "collocated" type output-feedback control system is proposed. The global exponential stability of the closed-loop system, boundedness, and exponential convergence of the controller, are proved via Lyapunov analysis. The performance is investigated via numerical simulation.**

***Index Terms*—Backstepping, boundary control, distributed parameter system, hyperbolic PDE.**

I. Introduction

A. Motivation

BRAKE performance is one of important safety indexes of a mining cable elevator [29], [30], where the brake system consists of a drum brake, a shock absorber, a time-varying-length cable, and a cage. In the process of stoppages, especially in the emergence stop of a high-speed elevator, the acceleration of the cage changes rapidly, which would cause large vibrations and significant oscillations of stress in the cable. It would not only lead to passengers discomfort or injured by impact [27], but also produce premature fatigue problems [33] which require frequent inspections, costly repairs, and may result in cable fracture in a worst case. Some researchers installed additional brake devices, such as a magnetorheological fluid damper [27] or a "safety gear" [19], between the cage and the rail guide to improve the brake performance of an elevator. It is a more economic and convenient way to design an appropriate brake control force applied at the drum brake to stop the cage smoothly through a shock absorber and a cable, without modifying the original structure of the mining cable elevator. This task can be mathematically abstracted as boundary control of coupled hyperbolic PDEs sandwiched between a cascade of a linear ODE and a nonlinear ODE on the actuated side and a linear ODE on the opposite side, with a PDE domain that is time-varying, for the dynamics of the aforementioned brake system.

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B. Control of Coupled Hyperbolic PDEs

A time-varying-length cable connecting with a cage is a major element in the cable elevator, which can be modeled as a coupled hyperbolic PDE-ODE system on a time-varying domain, obtained from transforming a wave PDE-ODE system with in-domain damping through Riemann transformation [33]. Control of the coupled hyperbolic PDEs has received much attention in the recent years. Some successful methods to stabilize coupled transport PDEs in different directions can be found in [7], [9], [15], [21], and [28], on the basis of which some adaptive control designs for the hyperbolic PDEs with uncertain system parameters were also proposed in [4], [6], [37], and [38]. Moreover, some results about boundary control of coupled transport PDE-ODE systems were presented in [1]–[3], [10], [12], [22], [26]. Boundary control designs of a class of infinite dimensional Port–Hamiltonian systems were proposed in [20] and [25]. The aforementioned papers focus on PDE systems on a fixed domain rather than a time-varying domain in accordance with the varying length of the cable. In a recent work [33], output-feedback control of a linear coupled hyperbolic PDE-ODE system on a time-varying domain was developed and applied into balancing control of a dual-cable mining cable elevator. However, all the above research considers control actuation directly flowing into the PDE boundary and ignoring the actuator dynamics. The dynamics of the drum brake and shock absorber have a significant influence on the brake performance of the mining cable elevator, so their dynamics, i.e., the actuator dynamics should be taken into consideration for the brake control design of the mining cable elevator, which produces a more challenging problem about control of a coupled hyperbolic PDE "sandwiched" system.

C. Control of "Sandwiched" PDE systems

Recently, some results about state-feedback and output-feedback control of coupled hyperbolic PDEs sandwiched between two ODEs were presented in [32] and [11], respectively, via the backstepping method. In addition to the coupled hyperbolic PDEs, control of transport PDE [5], [16], [17], viscous Burgers PDE [18] or heat PDE [35] sandwiched systems were also achieved successfully. However, these results only dealt with the problems where the PDE is on a fixed domain and sandwiched by linear ODEs. Control of a fixed-domain

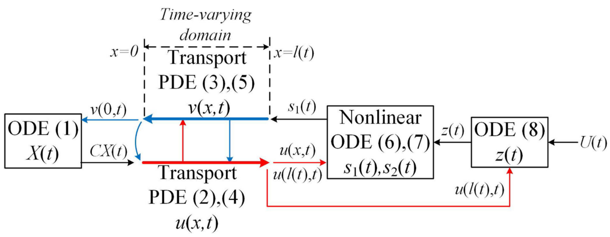

Fig. 1. Signal flow diagram of the plant (1)–(8).

sandwiched wave PDE including a nonlinear ODE was presented in [34] by using the naive and straightforward backstepping method, which results in some high-order derivative terms in the resulting control law. Motivated by brake control of a mining cable elevator mentioned in Section I-A, a more challenging task dealt with in this article is output-feedback control design of a particular class of coupled hyperbolic PDEs sandwiched between a cascade of a linear ODE and a nonlinear ODE on the actuated side and a linear ODE on the opposite side, with a PDE domain that is time-varying, as displayed in Fig. 1. Moreover, the control input should be guaranteed as boundedness and exponential convergence to zero.

D. Main Contribution

1) Compared with recent results on boundary control of ODE-PDE-ODE systems [5], [11], [23], we not only consider a PDE on a time-varying domain, but also deal with a cascade of ODEs with nonlinearity in the actuation path of the PDE, and the global exponential stability of the closed-loop system, the exponential convergence of the designed control input are achieved.
2) As compared to our previous result about state-feedback control of ODE-coupled hyperbolic PDE-ODE sandwiched system on a fixed domain [32], this article solves a more challenging problem where a nonlinear ODE exists in the input channel of the PDE into which it enters and which is on a time-varying domain. Moreover, a "collocated" type observer-based output-feedback controller without the derivatives of states is proposed.
3) This is the first result of stabilizing such a particular class of coupled hyperbolic PDEs sandwiched between a nonlinear ODE on the actuated side and a linear ODE on the opposite side, where the PDE domain is time-varying and the control action enters a single ODE state. Even if the time-varying domain is reduced to a fixed domain, the theoretical result is new.

E. Organization

The rest of the article is organized as follows. The problem formulation is presented in Section II. State-feedback control design and stability analysis are proposed in Section III. Observer design of the overall system is present and the exponential stability of the observer error system is proved in Section IV. The exponential stability of the output-feedback closed-loop system is given in Section V. The simulation results are provided in Section VI. The conclusion and future work are presented in Section VII.

Notation: Throughout this article, the partial derivatives and total derivatives are denoted as: $u_x(x,t) = \frac{\partial u}{\partial x}(x,t)$, $u_t(x,t) = \frac{\partial u}{\partial t}(x,t)$, $\gamma'(x) = \frac{d\gamma(x)}{dx}$, $\dot{X}(t) = \frac{dX(t)}{dt}$.

II. Problem Formulation

The plant considered in this article is

$$\dot{X}(t) = AX(t) + Bv(0,t) \tag{1}$$

$$u_t(x,t) = -p_1 u_x(x,t) + c_1 v(x,t) \tag{2}$$

$$v_t(x,t) = p_2 v_x(x,t) + c_2 u(x,t) \tag{3}$$

$$u(0,t) = qv(0,t) + CX(t) \tag{4}$$

$$v(l(t),t) = s_1(t) \tag{5}$$

$$\dot{s}_1(t) = c_3 s_2(t) + f_1\left(s_1(t), \int_0^{l(t)} u(x,t)dx\right) \tag{6}$$

$$\dot{s}_2(t) = f_2(s_1(t), s_2(t), u(l(t),t)) + z(t) \tag{7}$$

$$\dot{z}(t) = c_4 z(t) + ru(l(t),t) + U(t) \tag{8}$$

$\forall (x,t) \in [0, l(t)] \times [0, \infty)$, where $X(t) \in \mathbb{R}^{n\times 1}$, $z(t) \in \mathbb{R}$ are ODE states, which describe the vibration dynamics of the cage and drum. The nonlinear ODE-$S(t) = [s_1(t), s_2(t)]^T \in \mathbb{R}^{2\times 1}$ represents the shock absorber dynamics. $u(x,t) \in \mathbb{R}, v(x,t) \in \mathbb{R}$ are states of the 2×2 coupled hyperbolic PDEs, which model the vibration states of the cable. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ satisfy that the pair $[A, B]$ is controllable and $[A, C]$ is observable. $c_1, c_2, c_3, c_4, r, q \in \mathbb{R}$ are arbitrary. p_1 and p_2 are arbitrary positive transport velocities. $U(t)$ is the control input to be designed. The general nonlinear functions f_1 and f_2 can be unknown functions in the state-feedback control.

Functions f_1, f_2, and $l(t)$ satisfy the following assumptions.

Assumption 1: $f_1(0,0) = 0$ and $f_2(0,0,0) = 0$.

Assumption 2: $f_1(x_1,x_2)$ and $f_2(x_1,x_2,x_3)$ are continuously differentiable and globally Lipschitz in (x_1,x_2) and (x_1,x_2,x_3), respectively.

Assumption 3: $l(t) \in C^2(0,\infty)$. $l(t)$ is bounded: $0 < l(t) \le L, \forall t \ge 0$.

Assumption 4: Velocity $\dot{l}(t)$ of the moving boundary is bounded by

$$\left|\dot{l}(t)\right| < \min\{p_1, p_2\}. \tag{9}$$

Equations (1)–(5) can be regarded as being reversibly converted from a wave PDE with in-domain damping through the Riemann coordinate transformation [33]. Therefore, according to the conclusions in [13] and [14], the fact that the derivative of the moving boundary $l(t)$ is smaller than the wave speed, i.e., Assumption 4, allows to prove a well-posedness result for the initial boundary value problem (1)–(5).

The signal flow of the plant (1)–(8) is shown in Fig. 1, where the control input $U(t)$ goes through a linear ODE (8) acting as a filter, of which the output signal $z(t)$ drives a nonlinear ODE (6), (7) including states $s_2(t)$ and $s_1(t)$ which flows into the right boundary $x = l(t)$ (5) of the transport PDE-v (3), which is coupled with another transport PDE-u (2), (4) and connected with a linear ODE (1) at the left boundary $x = 0$. The reflect signals flow back to the ODEs (6)–(8) via the transport PDE-v.

The control objective and available measurements: the control objective here is to exponentially stabilize all ODE states $S(t), X(t), z(t)$ and PDE states $u(x,t), v(x,t)$ by designing a control input $U(t)$ applied at the first ODE (8), using the measurements $v(l(t),t), u(l(t),t), z(t)$.

III. State-Feedback Control Design

In Section III-A, a PDE backstepping transformation is used to convert the coupled hyperbolic PDE-ODE subsystem to a "stablelike" intermediate system where the in-domain couplings between the hyperbolic PDEs are removed and the state matrix of the ODE at the left boundary is Hurwitz. The right boundary condition of the intermediate system can be regarded as a cascade of a nonlinear ODE and a linear ODE under some perturbations from the PDE states in the time-varying domain and the left boundary, which would be dealt with by using an ODE backstepping procedure in Section III-B. The global exponential stability of the closed-loop system is proved in Section III-C, where some control parameters which come from the ODE backstepping design procedure and should tolerate the PDE perturbations are determined. Moreover, the boundedness and exponential convergence of the designed control input are proved as well.

A. Backstepping Transformation for PDE-ODE Subsystem $(u(x,t),\ v(x,t),\ X(t))$

We consider the infinite-dimensional backstepping transformation [32] of the PDE states $u(x,t), v(x,t)$

$$\alpha(x,t) = u(x,t) \tag{10}$$

$$\beta(x,t) = v(x,t) - \int_0^x \psi(x,y)u(y,t)dy - \int_0^x \phi(x,y)v(y,t)dy - \gamma(x)X(t). \tag{11}$$

The kernels $\psi(x,y), \phi(x,y)$ on $\mathcal{D} = \{0 \le y \le x \le l(t)\}$ and the row vector $\gamma(x)$ on $\{0 \le x \le l(t)\}$ satisfy

$$c_2 + (p_1 + p_2)\psi(x,x) = 0 \tag{12}$$

$$p_2\phi(x,0) = \gamma(x)B + p_1 q\psi(x,0) \tag{13}$$

$$-c_1\psi(x,y) + p_2\phi_x(x,y) + p_2\phi_y(x,y) = 0 \tag{14}$$

$$c_2\phi(x,y) - p_2\psi_x(x,y) + p_1\psi_y(x,y) = 0 \tag{15}$$

$$p_2\gamma'(x) - \gamma(x)A - p_1\psi(x,0)C = 0 \tag{16}$$

$$\gamma(0) = \kappa \tag{17}$$

where κ is a row vector such that $A + B\kappa$ is Hurwitz, since the pair $[A, B]$ is controllable. Please refer to Lemma 1 of [32] and Lemma 1 of [36] for the well posedness of (12)–(17).

The inverse of (10) and (11) is considered as

$$u(x,t) = \alpha(x,t) \tag{18}$$

$$v(x,t) = \beta(x,t) - \int_0^x \mathcal{D}(x,y)\alpha(y,t)dy - \int_0^x \mathcal{M}(x,y)\beta(y,t)dy - \mathcal{J}(x)X(t) \tag{19}$$

where $\mathcal{D}(x,y), \mathcal{M}(x,y)$, and the row vector $\mathcal{J}(x)$ are the kernels of the inverse transformation (19), of which the well posedness is shown in [32, Sec. 2.4].

Applying the above backstepping transformations, the original system (1)–(5) is converted to the following intermediate system (without the right boundary condition):

$$\dot{X}(t) = (A + B\kappa)X(t) + B\beta(0,t) \tag{20}$$

$$\alpha_t(x,t) = -p_1\alpha_x(x,t) + c_1\beta(x,t) - c_1\int_0^x \mathcal{D}(x,y)\alpha(y,t)dy - c_1\int_0^x \mathcal{M}(x,y)\beta(y,t)dy - c_1\mathcal{J}(x)X(t) \tag{21}$$

$$\beta_t(x,t) = p_2\beta_x(x,t) \tag{22}$$

$$\alpha(0,t) = q\beta(0,t) + C_0X(t) \tag{23}$$

where the row vector $C_0 = C + q\gamma(0)$. Let us now consider the right boundary condition. Inserting $x = l(t)$ into (11) and taking the derivative with respect to t, we have

$$\begin{aligned}\dot{\beta}(l(t),t) &= \dot{v}(l(t),t) - \dot{l}(t)\psi(l(t),l(t))u(l(t),t) \\ &- \dot{l}(t)\phi(l(t),l(t))v(l(t),t) - \dot{l}(t)\int_0^{l(t)} \psi_x(l(t),y)u(y,t)dy \\ &- \dot{l}(t)\int_0^{l(t)} \phi_x(l(t),y)v(y,t)dy \\ &- \dot{l}(t)\gamma'(l(t))X(t) - \int_0^{l(t)} \psi(l(t),y)u_t(y,t)dy \\ &- \int_0^{l(t)} \phi(l(t),y)v_t(y,t)dy - \gamma(l(t))\dot{X}(t).\end{aligned} \tag{24}$$

Using (5) and (6) to replace $\dot{v}(l(t),t)$ in (24), and then plugging the inverse transformations (18), (19) into (24) to replace u, v with α, β, through a change of the order of integration in a double integral, we get $\dot{\beta}(l(t),t)$ as

$$\begin{aligned}&\dot{\beta}(l(t),t) \\ &= c_3 s_2(t) + f_1\Big(\beta(l(t),t) - \int_0^{l(t)} \mathcal{D}(l(t),y)\alpha(y,t)dy \\ &- \int_0^{l(t)} \mathcal{M}(l(t),y)\beta(y,t)dy - \mathcal{J}(l(t))X(t), \int_0^{l(t)} \alpha(y,t)dy\Big) \\ &+ \mathcal{F}(\beta(l(t),\beta(0,t),\alpha(l(t),t),\alpha(0,t),\beta(x,t),\alpha(x,t),X(t))\end{aligned} \tag{25}$$

where $\mathcal{F}$ is a perturbation including $\beta(l(t),t), \beta(0,t), \alpha(l(t),t)$, $\alpha(0,t)$, $\beta(x,t)$, $\alpha(x,t)$, and $X(t)$. The complete expression of $\mathcal{F}$ is shown in Appendix-A. Recalling (5), (7), (8) and (18), (19), yields

$$\begin{aligned}\dot{s}_2(t) &= f_2\Big(\beta(l(t),t) - \int_0^{l(t)} \mathcal{D}(l(t),y)\alpha(y,t)dy \\ &- \int_0^{l(t)} \mathcal{M}(l(t),y)\beta(y,t)dy - \mathcal{J}(l(t))X(t), s_2(t), \alpha(l(t),t)\Big) \\ &+ z(t)\end{aligned} \tag{26}$$

$$\dot{z}(t) = c_4 z(t) + r\alpha(l(t),t) + U(t). \tag{27}$$

Note that (25)–(27) are the right boundary condition of the intermediate system in the form of several ODEs regulated by the control input $U(t)$.

Remark 1: (25)–(27) $(\beta(l(t),t),s_2(t),z(t))$ is a cascade of ODEs converted from (5)–(8) $(s_1(t),s_2,z(t))$ via transformation (10), (11). Equations (25) and (26) are a second-order nonlinear ODE $(\beta(l(t),t),s_2(t))$ with perturbations $\mathcal{F}$. Equation (27) is a first-order linear ODE $z(t)$ with a perturbation $\alpha(l(t),t)$.

Through the backstepping transformations (10), (11), the original system-$(u(x,t),v(x,t),X(t),s_1(t),s_2(t),z(t))$ (1)–(8) is converted to the intermediate system-$(\alpha(x,t)$, $\beta(x,t)$, $X(t)$, $\beta(l(t),t)$, $s_2(t)$, $z(t))$ (20)–(23), (25)–(27). Next, we propose backstepping design for the ODEs $(\beta(l(t),t),s_2(t),z(t))$ (25)–(27) at the right boundary of the intermediate system.

B. Backstepping Transformation for ODEs (25)–(27)

The following backstepping transformation for the $(\beta(l(t),t),s_2(t))$ system (25), (26) is made:

$$y_1(t)=\beta(l(t),t) \tag{28}$$

$$y_2(t)=s_2(t)+\tau_1(t) \tag{29}$$

where $\tau_1(t)$ to be defined in the following steps is the virtual control in the ODE backstepping method.

Step 1: We consider a Lyapunov function candidate as $V_{y1}=\frac{1}{2}y_1(t)^2$. Taking the derivative of V_{y1}, recalling (22), (25), and (29), we obtain

$$\begin{aligned}\dot V_{y1}&=y_1(t)\dot y_1(t)=y_1(t)\dot\beta(l(t),t)\\&=y_1(t)\left(c_3y_2(t)-c_3\tau_1(t)+f_1+\mathcal{F}\right).\end{aligned} \tag{30}$$

The arguments of f_1 and $\mathcal{F}$ are omitted in (30), which are the same as those in (25).

Define

$$\tau_1(t)=\frac{\bar c_1}{c_3}y_1(t) \tag{31}$$

where $\bar c_1$ is a positive constant to be determined later.

Substituting (31) into (30) yields

$$\dot V_{y1}=-\bar c_1y_1(t)^2+c_3y_1(t)y_2(t)+y_1(t)f_1+y_1(t)\mathcal{F}. \tag{32}$$

Step 2: A Lyapunov function candidate for $y_1(t),y_2(t)$ is considered as

$$V_y=V_{y1}+\frac{1}{2}y_2(t)^2=\frac{1}{2}y_1(t)^2+\frac{1}{2}y_2(t)^2. \tag{33}$$

Taking the derivative of (33), we have

$$\begin{aligned}\dot V_y=&-\bar c_1y_1(t)^2+c_3y_1(t)y_2(t)+y_1(t)f_1\\&+y_1(t)\mathcal{F}+y_2(t)\left(f_2+z(t)+\dot\tau_1\right)\end{aligned} \tag{34}$$

where (26) and (29) are used and the argument which is omitted of f_2 is same as that in (26).

Step.3: Define a new variable $\mathcal{E}(t)$ as

$$\mathcal{E}(t)=z(t)+\bar c_2y_2(t)+c_3y_1(t) \tag{35}$$

where the positive constant $\bar c_2$ is to be determined later.

Inserting (35) into (34) to replace $z(t)$, we have

$$\begin{aligned}\dot V_y=&-\bar c_1y_1(t)^2-\bar c_2y_2(t)^2+y_1(t)f_1+y_1(t)\mathcal{F}\\&+y_2(t)\mathcal{E}(t)+y_2(t)f_2+\frac{\bar c_1}{c_3}y_2(t)\dot y_1(t).\end{aligned} \tag{36}$$

Using (35), then (27) can be written as

$$\begin{aligned}\dot{\mathcal{E}}&=c_4\mathcal{E}(t)+r\alpha(l(t),t)+\bar c_2\dot y_2(t)+c_3\dot y_1(t)\\&\quad-c_4\bar c_2y_2(t)-c_4c_3y_1(t)+U(t).\end{aligned} \tag{37}$$

Choosing $U(t)$ in (37) as

$$U(t)=-\bar a_0\mathcal{E}(t)-r\alpha(l(t),t)+c_4\bar c_2y_2(t)+c_4c_3y_1(t) \tag{38}$$

we then have

$$\dot{\mathcal{E}}(t)=-k_{\mathcal{E}}\mathcal{E}(t)+\bar c_2\dot y_2(t)+c_3\dot y_1(t) \tag{39}$$

where $k_{\mathcal{E}}=\bar a_0-c_4>0$ by choosing the control gain $\bar a_0$.

Through the transformations (10), (11), (28), (29), and (35), the original system-$(u(x,t)$, $v(x,t)$, $X(t)$, $s_1(t)$, $s_2(t)$, $z(t))$ is converted to the target system-$(\alpha(x,t)$, $\beta(x,t)$, $X(t)$, $y_1(t)$, $y_2(t)$, $\mathcal{E}(t))$ where the ODE states and PDE states are coupled. The exponential stability of the target system will be clear in the following Lyapunov analysis via choosing control parameters $\bar c_1,\bar c_2,\bar a_0$.

C. Stability Analysis of State-Feedback Closed-Loop System

1) Controller: Substituting (10), (11), (28), (29), (31), (35) into (38), we get the controller expressed by the original states

$$\begin{aligned}U(t)&=-\bar a_0z(t)+(c_4-\bar a_0)\bar c_2s_2(t)-ru(l(t),t)\\&+(c_4-\bar a_0)\left(\frac{\bar c_1\bar c_2}{c_3}+c_3\right)\left(s_1(t)-\int_0^{l(t)}\psi(l(t),y)u(y,t)dy\right.\\&\left.-\int_0^{l(t)}\phi(l(t),y)v(y,t)dy-\gamma(l(t))X(t)\right).\end{aligned} \tag{40}$$

The pending control parameters $\bar c_1,\bar c_2,\bar a_0$ will be determined in the following stability analysis. Note that the control law (40) uses the signal $u(l(t),t)$. In order to ensure the control law is sufficiently regular, we will require the initial value $u(x,0)$ to be in $H^1(0,L)$ which is defined as $H^1(0,L)=\{u|u\in L^2(0,L),u_x\in L^2(0,L)\}$, where $L^2(0,L)$ is the usual Hilbert space and the positive constant L given in Assumption 3 is the maximum value of the time-varying PDE domain.

2) Stability of Closed-Loop System: *Theorem 1:* If initial values $(u(x,0),v(x,0))\in H^1(0,L)$, for some $\bar c_1,\bar c_2,\bar a_0$, the closed-loop system consisting of the plant (1)–(8) and the control law (40) is exponentially stable in the sense of that there exist positive constants Υ_1,λ_1 such that

$$\Omega_a(t)\le\Upsilon_1\Omega_a(0)e^{-\lambda_1t} \tag{41}$$

where

$$\begin{aligned}\Omega_a(t)=&\|u(\cdot,t)\|^2+\|v(\cdot,t)\|^2+|X(t)|^2+s_1(t)^2\\&+s_2(t)^2+z(t)^2.\end{aligned} \tag{42}$$

$\|u(\cdot,t)\|^2$ is a compact notation for $\int_0^{l(t)}u^2(x,t)dx$.

Proof: We start from studying the stability of the target system. The equivalent stability property between the target system and the original system is ensured due to the invertibility of the transformations (10), (11), (28), (29), and (35).

First, we study the stability proof of the target system via Lyapunov analysis of the PDE-ODE subsystem. Second, combining the Lyapunov analysis of ODEs in the input channel in Section III-B, Lyapunov analysis of the overall system is provided, where the control parameters $\bar c_1,\bar c_2,\bar a_0$ in the control law (40) are determined.

a) Lyapunov analysis for the PDE-ODE subsystem-$(\alpha(x,t),\beta(x,t),X(t))$: Consider now a Lyapunov function

$$\begin{aligned}V_1(t)=&X^T(t)P_1X(t)+\frac{a_1}{2}\int_0^{l(t)}e^{\delta_1x}\beta(x,t)^2dx\\&+\frac{b_1}{2}\int_0^{l(t)}e^{-\delta_1x}\alpha(x,t)^2dx\end{aligned} \tag{43}$$

where $P_1 = P_1^T > 0$ is the solution to the Lyapunov equation $P_1(A+B\kappa)+(A+B\kappa)^T P_1 = -Q_1$, for some $Q_1 = Q_1{}^T > 0$. The positive parameters a_1, b_1, δ_1 are to be chosen later.

Taking the derivative of $V_1(t)$, we arrive at

$$\dot{V}_1(t) \le -\eta_1|X(t)|^2 - \eta_2\beta(0,t)^2 - \eta_3\int_0^{l(t)}\beta(x,t)^2dx - \eta_4\int_0^{l(t)}\alpha(x,t)^2dx - \eta_5\alpha(l(t),t)^2 + \eta_6\beta(l(t),t)^2 \quad (44)$$

where the detailed process of calculating $\dot{V}_1(t)$ are shown in Appendix-B, where the choices of a_1, b_1, δ_1 and the expressions of positive constants η_1, η_2, η_3, η_4 are also given. Defining $v_{max} = \max_{t\in[0,\infty)}\{|\dot{l}(t)|\}$, we know $\eta_5 = (p_1 - v_{max})\frac{b_1}{2}e^{-\delta_1 L} > 0$ by recalling Assumption 4, and $\eta_6 = (p_2 + v_{max})\frac{a_1}{2}e^{\delta_1 L} > 0$.

b) Lyapunov analysis for the overall system: Consider a Lyapunov function as

$$V(t) = V_1(t) + V_y(t) + \frac{1}{2}\mathcal{E}(t)^2. \quad (45)$$

Defining

$$\Omega_1(t) = \|\beta(\cdot,t)\|^2 + \|\alpha(\cdot,t)\|^2 + |X(t)|^2 + y_1(t)^2 + y_2(t)^2 + \mathcal{E}(t)^2 \quad (46)$$

we have

$$\theta_{1a}\Omega_1(t) \le V(t) \le \theta_{1b}\Omega_1(t) \quad (47)$$

for some positive constants θ_{1a} and θ_{1b}.

Taking the derivative of (45), using (36), (39), and (44) with (A.1)–(A.8), recalling Assumptions 1, 2, 4, we have

$$\dot{V}(t) \le -\lambda V(t) - \hat{\eta}_0\beta(0,t)^2 - \hat{\eta}_1\alpha(l(t),t)^2 \quad (48)$$

for some positive λ, and $\hat{\eta}_0$ $\hat{\eta}_1$ are positive constants given as (C.12)–(C.13). The detailed process of calculating $\dot{V}(t)$ is shown in Appendix-C, where the choices of the control parameters $\bar{c}_1, \bar{c}_2$, a_0 in the ODE backstepping to tolerate the PDE perturbations are presented.

We thus have

$$V(t) \le V(0)e^{-\lambda t}. \quad (49)$$

It then follows that $\Omega_1(t) \le \frac{\theta_{1b}}{\theta_{1a}}\Omega_1(0)e^{-\lambda t}$ by recalling (47).

Defining

$$\Xi(t) = \|u(\cdot,t)\|^2 + \|v(\cdot,t)\|^2 + |X(t)|^2 + s_1(t)^2 + s_2(t)^2 + z(t)^2 \quad (50)$$

applying Cauchy–Schwarz inequality and transformations (10), (11), (18), (19), (28), (29), and (35), it is straightforward to obtain

$$\bar{\theta}_{1a}\Xi(t) \le \Omega_1(t) \le \bar{\theta}_{1b}\Xi(t) \quad (51)$$

for some positive $\bar{\theta}_{1a}$ and $\bar{\theta}_{1b}$. Therefore, we have

$$\Xi(t) \le \frac{\theta_{1b}\bar{\theta}_{1b}}{\theta_{1a}\bar{\theta}_{1a}}\Xi(0)e^{-\lambda t}. \quad (52)$$

Thus, (41) is achieved with

$$\Upsilon_1 = \frac{\theta_{1b}\bar{\theta}_{1b}}{\theta_{1a}\bar{\theta}_{1a}},\ \lambda_1 = \lambda. \quad (53)$$

Then the proof of Theorem 1 is completed. ■

3) Exponential Convergence of Control Input: In Theorem 1, we have proved that all PDEs and ODEs are exponentially stable in the closed-loop system including the plant (1)–(8) and the controller (40). Moreover, next we would prove the controller $U(t)$ (40) in the closed-loop system is also bounded and exponentially convergent to zero.

Considering (40) and the exponential stability result proved in Theorem 1, the exponential convergence of the control input requires the exponential convergence of the signal $u(l(t),t)$ additionally, which can be obtained by proving the exponential stability estimate of $\|u_x(\cdot,t)\| + \|v_x(\cdot,t)\|$. Before proving the exponential convergence of the control input, we propose a lemma first.

Lemma 1: For any initial data $(u(x,0), v(x,0)) \in H^1(0,L)$, the exponential stability estimate of the closed-loop system $(u(x,t), v(x,t))$ is obtained in the sense of that there exist positive constants Υ_{1a} and λ_{1a} such that

$$\|u_x(\cdot,t)\|^2 + \|v_x(\cdot,t)\|^2 \le \Upsilon_{1a}\left(\Xi(0) + \|u_x(\cdot,0)\|^2 + \|v_x(\cdot,0)\|^2\right)e^{-\lambda_{1a}t} \quad (54)$$

where $\Xi(t)$ is given in (50).

The proof of Lemma 1 is shown in Appendix-D. Lemma 1 will be used in proving the exponential convergence and boundedness of the controller (40) in the following theorem.

Theorem 2: In the closed-loop system including the plant (1)–(8) and the controller $U(t)$ (40), there exist positive constants λ_2 and Υ_2 making that $U(t)$ is bounded and exponentially convergent to zero in the sense of

$$|U(t)| \le \Upsilon_2(\|u(\cdot,0)\|^2 + \|v(\cdot,0)\|^2 + |X(0)|^2 + s_1(0)^2 + s_2(0)^2 + z(0)^2 + \|u_x(\cdot,0)\|^2 + \|v_x(\cdot,0)\|^2)^{\frac{1}{2}}e^{-\lambda_2 t}. \quad (55)$$

Proof: The proof is shown in Appendix-E. ■

IV. Observer Design and Stability Analysis

In Section III, a state-feedback controller which requires distributed states is designed to stabilize the original system exponentially. However, it is always difficult to measure the distributed states in practice. We propose an output-feedback control law which only requires measurements $u(l(t),t), v(l(t),t), z(t)$ at the controlled boundary of the PDE, i.e., a "collocated" type, based on a state observer designed in this section. In Section IV-A, the observer design is presented, where the observer gains are determined in two transformation processes from the observer error system to an intermediate observer error system, and then to a target observer error system. The exponential stability of the observer error system is proved in Section IV-B.

A. Observer Design

1) Structure of Observer and Error Dynamics: Using the measurements $u(l(t),t), v(l(t),t), z(t)$, the observer is designed as

$$\dot{\hat{X}}(t) = A\hat{X}(t) + B\hat{v}(0,t) + \Gamma_0(t)(u(l(t),t) - \hat{u}(l(t),t)) \quad (56)$$

$$\hat{u}_t(x,t) = -p_1\hat{u}_x(x,t) + c_1\hat{v}(x,t) + \Gamma_1(x,t)(u(l(t),t) - \hat{u}(l(t),t)) \quad (57)$$

$$\hat{v}_t(x,t) = p_2\hat{v}_x(x,t) + c_2\hat{u}(x,t)$$

$$+\Gamma_2(x,t)(u(l(t),t)-\hat{u}(l(t),t)) \quad (58)$$

$$\hat{u}(0,t)=q\hat{v}(0,t)+C\hat{X}(t) \quad (59)$$

$$\hat{v}(l(t),t)=v(l(t),t) \quad (60)$$

$$\dot{\hat{s}}_1(t)=c_3\hat{s}_2(t)+f_1\left(\hat{s}_1(t),\int_0^{l(t)}\hat{u}(y,t)dy\right) + \mu_1(v(l(t),t)-\hat{s}_1(t))) \quad (61)$$

$$\dot{\hat{s}}_2(t)=f_2(\hat{s}_1(t),\hat{s}_2(t),\hat{u}(l(t),t))+z(t) + \mu_2(v(l(t),t)-\hat{s}_1(t))) \quad (62)$$

$$\dot{\hat{z}}(t)=c_4\hat{z}(t)+ru(l(t),t)+\mu_3(z(t)-\hat{z}(t))+U(t) \quad (63)$$

where $\Gamma_0(t),\Gamma_1(x,t),\Gamma_2(x,t),\mu_1,\mu_2,\mu_3$ are observer gains to be determined later. Note that the initial values $\hat{u}(x,0),\hat{v}(x,0)$ are required to be in $H_1(0,L)$ to be consistent with Section III. Define observer errors as

$$[\tilde{X}(t),\tilde{u}(x,t),\tilde{v}(x,t),\tilde{s}_1(t),\tilde{s}_2(t),\tilde{z}(t)] = [X(t),u(x,t),v(x,t),s_1(t),s_2(t),z(t)] - [\hat{X}(t),\hat{u}(x,t),\hat{v}(x,t),\hat{s}_1(t),\hat{s}_2(t),\hat{z}(t)]. \quad (64)$$

According to (56)–(63) and (1)–(8), the error dynamics can be obtained as

$$\dot{\tilde{X}}(t)=A\tilde{X}(t)+B\tilde{v}(0,t)-\Gamma_0(t)\tilde{u}(l(t),t) \quad (65)$$

$$\tilde{u}_t(x,t)=-p_1\tilde{u}_x(x,t)+c_1\tilde{v}(x,t)-\Gamma_1(x,t)\tilde{u}(l(t),t) \quad (66)$$

$$\tilde{v}_t(x,t)=p_2\tilde{v}_x(x,t)+c_2\tilde{u}(x,t)-\Gamma_2(x,t)\tilde{u}(l(t),t) \quad (67)$$

$$\tilde{u}(0,t)=q\tilde{v}(0,t)+C\tilde{X}(t) \quad (68)$$

$$\tilde{v}(l(t),t)=0 \quad (69)$$

$$\dot{\tilde{s}}_1(t)=c_3\tilde{s}_2(t)+\tilde{f}_1-\mu_1\tilde{s}_1(t) \quad (70)$$

$$\dot{\tilde{s}}_2(t)=\tilde{f}_2-\mu_2\tilde{s}_1(t) \quad (71)$$

$$\dot{\tilde{z}}(t)=-k_z\tilde{z}(t) \quad (72)$$

where $k_z=\mu_3-c_4>0$ by choosing the control parameter μ_3, and

$$\tilde{f}_1=f_1\left(s_1(t),\int_0^{l(t)}u(y,t)dy\right) - f_1\left(\hat{s}_1(t),\int_0^{l(t)}\hat{u}(y,t)dy\right) \quad (73)$$

$$\tilde{f}_2=f_2\left(s_1(t),s_2(t),u(l(t),t)\right) - f_2\left(\hat{s}_1(t),\hat{s}_2(t),\hat{u}(l(t),t)\right). \quad (74)$$

Defining

$$\tilde{S}(t)=[\tilde{s}_1(t),\tilde{s}_2(t)]^T \quad (75)$$

(70), (71) can be rewritten as

$$\dot{\tilde{S}}(t)=(A_s-\mathcal{B}C_2)\tilde{S}(t)+\left[\tilde{f}_1,\tilde{f}_2\right]^T \quad (76)$$

where

$$A_s=\begin{pmatrix}0 & c_3\\ 0 & 0\end{pmatrix},\quad C_2=[1,0],\quad \mathcal{B}=[\mu_1,\mu_2]^T. \quad (77)$$

Note that $A_s-\mathcal{B}C_2$ can be a Hurwitz matrix by choosing $\mathcal{B}=[\mu_1,\mu_2]^T$, because (A_s,C_2) is observable.

2) Transformation to Intermediate Observer Error System: In order to remove domain couplings in $\tilde{v}$ (67) which affect the system stability [33], we apply the invertible backstepping transformation [33] for the PDE states $(\tilde{u},\tilde{v})$

$$\tilde{u}(x,t)=\tilde{\alpha}(x,t)-\int_x^{l(t)}\bar{\phi}(x,y)\tilde{\alpha}(y,t)dy \quad (78)$$

$$\tilde{v}(x,t)=\tilde{\beta}(x,t)-\int_x^{l(t)}\bar{\psi}(x,y)\tilde{\alpha}(y,t)dy \quad (79)$$

to convert the error dynamics (65)–(72) to the intermediate observer error system as

$$\dot{\tilde{X}}(t)=A\tilde{X}(t)+B\tilde{\beta}(0,t)-B\int_0^{l(t)}\bar{\psi}(0,y)\tilde{\alpha}(y,t)dy - \Gamma_0(t)\tilde{\alpha}(l(t),t) \quad (80)$$

$$\tilde{\alpha}_t(x,t)=-p_1\tilde{\alpha}_x(x,t)+\int_x^{l(t)}\bar{M}(x,y)\tilde{\beta}(y,t)dy + c_1\tilde{\beta}(x,t) \quad (81)$$

$$\tilde{\beta}_t(x,t)=p_2\tilde{\beta}_x(x,t)+\int_x^{l(t)}\bar{N}(x,y)\tilde{\beta}(y,t)dy \quad (82)$$

$$\tilde{\alpha}(0,t)=q\tilde{\beta}(0,t)+C\tilde{X}(t) + \int_0^{l(t)}(\bar{\phi}(0,y)-q\bar{\psi}(0,y))\tilde{\alpha}(y,t)dy \quad (83)$$

$$\tilde{\beta}(l(t),t)=0. \quad (84)$$

$$\dot{\tilde{S}}(t)=(A_s-\mathcal{B}C_2)\tilde{S}(t)+[\tilde{f}_1,\tilde{f}_2]^T \quad (85)$$

$$\dot{\tilde{z}}(t)=-k_z\tilde{z}(t). \quad (86)$$

By matching (65)–(69) and (80)–(84), the kernel functions $\bar{\phi},\bar{\psi}$ on $\mathcal{D}_1=\{0\le x\le y\le l(t)\}$ should satisfy

$$-p_1\bar{\phi}_x(x,y)-p_1\bar{\phi}_y(x,y)-c_1\bar{\psi}(x,y)=0 \quad (87)$$

$$\bar{\psi}(x,x)=\frac{c_2}{p_1+p_2} \quad (88)$$

$$-p_1\bar{\psi}_y(x,y)+p_2\bar{\psi}_x(x,y)-c_2\bar{\phi}(x,y)=0. \quad (89)$$

The boundary condition of $\bar{\phi}$ is set as

$$\bar{\phi}(0,y)=q\bar{\psi}(0,y)-CK_0(y) \quad (90)$$

where $K_0(x)$ is shown later. The choice of (90) would be clear later.

$\bar{M}(x,y),\bar{N}(x,y)$ in (80)–(86) satisfy

$$\bar{M}(x,y)=\int_x^y\bar{\phi}(x,z)\bar{M}(z,y)dz-c_1\bar{\phi}(x,y) \quad (91)$$

$$\bar{N}(x,y)=\int_x^y\bar{\psi}(x,z)\bar{M}(z,y)dz+c_1\bar{\psi}(x,y). \quad (92)$$

Observer gains $\Gamma_1(x,t)$ and $\Gamma_2(x,t)$ are obtained as

$$\Gamma_1(x,t)=\dot{l}(t)\bar{\phi}(x,l(t))-p_1\bar{\phi}(x,l(t)) \quad (93)$$

$$\Gamma_2(x,t)=\dot{l}(t)\bar{\psi}(x,l(t))-p_1\bar{\psi}(x,l(t)). \quad (94)$$

3) Transformation to Target Observer Error System: In order to decouple the ODE (80) with the PDE state-$\tilde{\alpha}$ ($\tilde{\beta}$ reaches to zero after a finite time because of (84), (82)) and make the state matrix in the ODE (80) Hurwitz, where the observer gain $\Gamma_0(t)$ would be defined, we apply a transformation as

$$\tilde{Y}(t) = \tilde{X}(t) - \int_0^{l(t)} K_0(x)\tilde{\alpha}(x,t)dx - \int_0^{l(t)} K_1(x)\tilde{\beta}(x,t)dx \tag{95}$$

to convert (80) into

$$\dot{\tilde{Y}}(t) = (A - L_0C)\tilde{Y}(t) - \int_0^{l(t)} \left[\int_0^x K_0(y)\bar{M}(y,x)dy + \int_0^x K_1(y)\bar{N}(y,x)dy\right]\tilde{\beta}(x,t)dx \tag{96}$$

where $A - L_0C$ is a Hurwitz matrix by choosing L_0 recalling that (A, C) is observable, and $K_0(x)$, $K_1(x)$ are determined following.

Substituting (95) into (96), considering (80)–(84), using integration by parts and a change of the order of integration in a double integral, we have

$$\begin{aligned}
&\left[K_0(l(t))p_1 - \dot{l}(t)K_0(l(t)) - \Gamma_0(t)\right]\tilde{\alpha}(l(t),t)\\
&- \int_0^{l(t)} \left[K_0'(x)p_1 - AK_0(x) + B\bar{\psi}(0,x)\right]\tilde{\alpha}(x,t)dx\\
&+ (L_0 - K_0(0)p_1)\tilde{\alpha}(0,t) + \int_0^{l(t)} [-K_0(x)c_1 + K_1'(x)p_2\\
&+(A - L_0C)K_1(x)]\tilde{\beta}(x,t)dx\\
&+ [K_1(0)p_2 - L_0q + B]\tilde{\beta}(0,t) = 0.
\end{aligned} \tag{97}$$

For (97) to hold, $K_0(x)$, $K_1(x)$ should satisfy

$$K_0'(x)p_1 - AK_0(x) + B\bar{\psi}(0,x) = 0 \tag{98}$$

$$K_0(0) = \frac{L_0}{p_1} \tag{99}$$

$$K_1'(x)p_2 + (A - L_0C)K_1(x) - K_0(x)c_1 = 0 \tag{100}$$

$$K_1(0) = \frac{L_0q - B}{p_2}. \tag{101}$$

Lemma 2: Equations (87)–(90), (98)–(101) of conditions of kernels $\bar{\phi}(x,y), \bar{\psi}(x,y), K_0(x), K_1(x)$ are well-posed.

Proof: After swapping positions of arguments as B.9-B.10 in [3], i.e., changing the domain $\mathcal{D}_1$ to $\mathcal{D}$, conditions (87)–(90), (98), (99) of $\bar{\phi}, \bar{\psi}, K_0$ have the same form of the conditions (12)–(17) on the kernels ϕ, ψ, γ which have been proved as well-posed in [32] and [36]. The explicit solutions of $K_1(x)$ are then easy to obtain considering the initial value problem (100), (101). ■

The observer gain $\Gamma_0(t)$ is obtained as

$$\Gamma_0(t) = -\dot{l}(t)K_0(l(t)) + K_0(l(t))p_1. \tag{102}$$

The target observer error system thus can be written as

$$\dot{\tilde{Y}}(t) = (A - L_0C)\tilde{Y}(t) - \int_0^{l(t)} \left[\int_0^x K_0(y)\bar{M}(y,x)dy + \int_0^x K_1(y)\bar{N}(y,x)dy\right]\tilde{\beta}(x,t)dx \tag{103}$$

$$\tilde{\alpha}_t(x,t) = -p_1\tilde{\alpha}_x(x,t) + \int_x^{l(t)} \bar{M}(x,y)\tilde{\beta}(y,t)dy + c_1\tilde{\beta}(x,t) \tag{104}$$

$$\tilde{\beta}_t(x,t) = p_2\tilde{\beta}_x(x,t) + \int_x^{l(t)} \bar{N}(x,y)\tilde{\beta}(y,t)dy \tag{105}$$

$$\tilde{\alpha}(0,t) = q\tilde{\beta}(0,t) + C\tilde{Y}(t) + \int_0^{l(t)} CK_1(y)\tilde{\beta}(y,t)dy \tag{106}$$

$$\tilde{\beta}(l(t),t) = 0. \tag{107}$$

$$\dot{\tilde{S}}(t) = (A_s - \mathcal{B}C_2)\tilde{S}(t) + \left[\tilde{f}_1, \tilde{f}_2\right]^T \tag{108}$$

$$\dot{\tilde{z}}(t) = -k_z\tilde{z}(t). \tag{109}$$

The following theorem shows the exponential stability of the observer error system (65)–(72), which is obtained through the stability analysis of the target observer error system (103)–(109) and applying the invertibility of the transformations. Note that the initial data $(\tilde{u}(x,0), \tilde{v}(x,0))$ of the observer error system belongs to $H^1(0,L)$, which is defined by the initial conditions of the plant and the observer via (64).

B. Stability Analysis of Observer Error System

Theorem 3: Considering the observer system (56)–(63) with observer gains $\Gamma_0(t)$ (102), $\Gamma_1(x,t)$ (93), $\Gamma_2(x,t)$ (94), the observer error system (65)–(72) is exponentially stable in the sense of that there exist positive constants Υ_e, λ_e such that

$$\Omega_e(t) \le \Upsilon_e\Omega_e(0)e^{-\lambda_e t} \tag{110}$$

where

$$\Omega_e(t) = \|\tilde{u}(\cdot,t)\|^2 + \|\tilde{v}(\cdot,t)\|^2 + \left|\tilde{X}(t)\right|^2 + \tilde{s}_1(t)^2 + \tilde{s}_2(t)^2 + \tilde{z}(t)^2. \tag{111}$$

Proof: a) Analysis for the observer error subsystems- $(\tilde{u}(x,t), \tilde{v}(x,t), \tilde{X}(t), \tilde{z}(t))$: (108) $\tilde{z}(t)$ is an exponentially stable ODE because of $k_z > 0$. From (105), (107), the $\tilde{\beta}$-dynamics is independent of $\tilde{\alpha}$ and $\tilde{\beta}(x,t) \equiv 0$ after $t_{f0} = \frac{L}{p_2}$, i.e., when the boundary condition (107) has propagated through the whole domain. The subsystem (103)–(107) becomes

$$\dot{\tilde{Y}}(t) = (A - L_0C)\tilde{Y}(t) \tag{112}$$

$$\tilde{\alpha}_t(x,t) = -p_1\tilde{\alpha}_x(x,t) \tag{113}$$

$$\tilde{\alpha}(0,t) = C\tilde{Y}(t) \tag{114}$$

for $t \ge t_{f0}$. $\tilde{Y}(t)$ is exponentially convergent to zero because $A - L_0C$ in the ODE (112) is Hurwitz. Define

$$V_a(t) = \tilde{Y}(t)^T P_a\tilde{Y}(t) + \frac{b_a}{2}\int_0^{l(t)} e^{-x}\tilde{\alpha}(x,t)^2dx \tag{115}$$

where b_a is a positive constant, and $P_a = P_a^T > 0$ is the solution to the Lyapunov equation $P_a(A - L_0C) + (A - L_0C)^T P_a = -Q_a$ for some $Q_a = Q_a{}^T > 0$.

Taking the derivative of $V_a(t)$ along (112)–(114), we have

$$\dot{V}_a(t) \le -\lambda_{\min}(Q_a)\tilde{Y}(t)^2 - p_1b_a\int_0^{l(t)} e^{-x}\tilde{\alpha}(x,t)\tilde{\alpha}_x(x,t)dx$$

$$
\begin{aligned}
&+\frac{b_a}{2}\dot{l}(t)e^{-l(t)}\tilde{\alpha}((t),t)^2\\
&\le -\left(\lambda_{\min}(Q_a)-\frac{1}{2}p_1 b_a|C|^2\right)\tilde{Y}(t)^2\\
&-\frac{1}{2}b_a(p_1-\dot{l}(t))e^{-l(t)}\tilde{\alpha}(l(t),t)^2\\
&-\frac{1}{2}p_1 b_a\int_0^{l(t)} e^{-x}\tilde{\alpha}(x,t)^2dx. \qquad (116)
\end{aligned}
$$

Choosing $b_a < \frac{2\lambda_{\min}(Q_a)}{p_1|C|^2}$ and recalling Assumption 4, yields

$$\dot{V}_a(t) \le -\lambda_a V_a(t) - \lambda_{a1}\tilde{\alpha}(l(t),t)^2 \qquad (117)$$

for some positive λ_a, λ_{a1}. The exponential stability result in the sense of $|\tilde{Y}(t)|^2 + \|\tilde{\alpha}(\cdot,t)\|^2 + \|\tilde{\beta}(\cdot,t)\|^2$ is obtained.

Remark 2: Even though $\tilde{\beta}(x,t) \equiv 0$ and (112)–(114) holds for $t \ge t_{f0}$ (if $\tilde{\beta}(0,0) = 0$, they hold at $t = 0$, and the obtained exponential stability straightforwardly begins from $t = 0$), the obtained exponential stability also holds at the beginning $t = 0$, because any transient in the finite time $[0, t_{f0})$ can be bounded by an exponentially decay signal with arbitrary decay rate and an appropriate overshoot coefficient.

According to the invertible transformation (78)–(79), (95), we obtain the exponential stability in the sense of $|\tilde{X}(t)|^2 + \|\tilde{u}(\cdot,t)\|^2+\|\tilde{v}(\cdot,t)\|^2$.

b) Analysis for the observer error subsystem-$\tilde{S}(t)$: Next we conduct the stability analysis for the ODE-$\tilde{S}(t)$ (108). Consider a Lyapunov function

$$V_s(t) = \tilde{S}(t)^T P_0 \tilde{S}(t) \qquad (118)$$

where P_0 is a positive definite and symmetric solution of

$$(A_s - \mathcal{B}C_2)^T P_0 + P_0(A_s - \mathcal{B}C_2) + \bar{\gamma}^2 P_0^T P_0 + I^T < 0 \qquad (119)$$

with $\bar{\gamma}^2 = \gamma_1^2 + 2\gamma_2^2$ and γ_1, γ_2 being positive Lipschitz constants shown in Appendix-F. The existence of the solution P_0 of (119) and the procedure to define the observer gain $\mathcal{B}$ are shown in [24, Sec. 4].

Taking the derivative of $V_s(t)$ (118), through the calculation process presented in Appendix-F where Assumption 2 is recalled, we achieve

$$
\begin{aligned}
\dot{V}_s(t) \le& -\sigma_s V_s(t) + \frac{\hat{\gamma}_1^2}{\gamma_1^2+2\gamma_2^2}\|\tilde{\alpha}(\cdot,t)\|^2\\
&+\frac{\gamma_2^2}{\gamma_1^2+2\gamma_2^2}\tilde{\alpha}(l(t),t)^2 \qquad (120)
\end{aligned}
$$

for some positive σ_s.

Consider a Lyapunov function

$$V_e(t) = V_s(t) + R_{\tilde{\alpha}}V_a(t). \qquad (121)$$

Taking the derivative of (121), recalling (120), (117), and choosing large enough positive constant $R_{\tilde{\alpha}}$, we have

$$
\begin{aligned}
\dot{V}_e(t) \le& -\sigma_s V_s(t) - \frac{1}{2}R_{\tilde{\alpha}}\lambda_a V_a(t)\\
&-\left(\frac{1}{4}R_{\tilde{\alpha}}\lambda_a b_a e^{-L} - \frac{\hat{\gamma}_1^2}{\gamma_1^2+2\gamma_2^2}\right)\|\tilde{\alpha}(\cdot,t)\|^2\\
&-\left(R_{\tilde{\alpha}}\lambda_{a1} - \frac{\gamma_2^2}{\gamma_1^2+2\gamma_2^2}\right)\tilde{\alpha}(l(t),t)^2\\
\le& -\sigma_e V_e(t) \qquad (122)
\end{aligned}
$$

for some positive σ_e. Considering (121), (118), we obtain $|\tilde{S}(t)|^2$ is exponentially convergent to zero.

Finally, using the obtained exponential stability result in the sense of $\|\tilde{u}(\cdot,t)\|^2 + \|\tilde{v}(\cdot,t)\|^2 + |\tilde{X}(t)|^2 + |\tilde{z}(t)|^2$ in 1) and the exponential stability result of $|\tilde{S}(t)|^2$ in 2) with (75), the proof of Theorem 3 is completed. ■

Moreover, we propose the following lemma which shows the exponential stability estimates in the sense of $\|\tilde{v}_x(\cdot,t)\|^2 + \|\tilde{u}_x(\cdot,t)\|^2$.

Lemma 3: The exponential stability estimate of the observer error system $(\tilde{u}(x,t), \tilde{v}(x,t))$ is obtained in the sense of that there exist positive constants $\Upsilon_{3a}, \lambda_{3a}$ such that

$$
\begin{aligned}
&\|\tilde{v}_x(\cdot,t)\|^2 + \|\tilde{u}_x(\cdot,t)\|^2\\
&\le \Upsilon_{3a}\left(\Omega_e(t) + \|\tilde{u}_x(\cdot,0)\|^2 + \|\tilde{v}_x(\cdot,0)\|^2\right)e^{-\lambda_{3a}t}.
\end{aligned}
$$

Proof: Take the spatial derivative of (104), (105), and the time derivative of (103), (106), (107), where $\dot{\tilde{\beta}}(l(t),t) = \dot{l}(t)\tilde{\beta}_x(l(t),t) + \tilde{\beta}_t(l(t),t) = (\dot{l}(t)+p_2)\tilde{\beta}_x(l(t),t) = 0$ is used, which results in $\tilde{\beta}_x(l(t),t) = 0$ because of $\dot{l}(t) + p_2 \ne 0$ recalling Assumption 4. Through the similar steps in the section a) of the proof of Theorem 3, using the spatial derivative of the transformation (78), (79), recalling Theorem 3, then Lemma 3 is obtained. ■

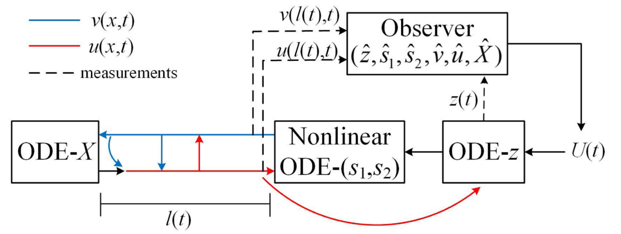

Fig. 2. Block diagram of output-feedback closed-loop system consisting of the plant (1)–(8), the observer (56)–(63), and the controller (123).

V. Stability of Output-Feedback Closed-Loop System

Replacing all the original states in the state-feedback controller (40) by the observer states, the output-feedback controller can be written as

$$
\begin{aligned}
&U_{of}(t) = -\bar{a}_0\hat{z}(t) + (c_4-\bar{a}_0)\bar{c}_2\hat{s}_2(t) - r\hat{u}(l(t),t)\\
&+(c_4-\bar{a}_0)\left(\frac{\bar{c}_1\bar{c}_2}{c_3}+c_3\right)\left(\hat{s}_1(t) - \int_0^{l(t)}\psi(l(t),y)\hat{u}(y,t)dy\right.\\
&\left.-\int_0^{l(t)}\phi(l(t),y)\hat{v}(y,t)dy - \gamma(l(t))\hat{X}(t)\right). \qquad (123)
\end{aligned}
$$

The output-feedback closed-loop system consists of the plant (1)–(8), the observer (56)–(63), and the output-feedback controller (123). The block diagram of the output-feedback closed-loop system is shown in Fig. 2. The following theorem shows the exponential stability of the output-feedback closed-loop system and the exponential convergence of the controller (123).

Theorem 4: Considering the plant (1)–(8), with the observer (56)–(63), and the output-feedback controller (123), initial values $(u(\cdot,0), v(\cdot,0)) \in H^1(0,L), (\hat{u}(\cdot,0), \hat{v}(\cdot,0)) \in H^1(0,L)$, the closed-loop system has the following properties.

1) There exist positive constants Υ_4 and λ_4 such that

$$\Omega(t) \leq \Upsilon_4 \Omega(0) e^{-\lambda_4 t} \tag{124}$$

where

$$\Omega(t) = \|\hat{v}(\cdot,t)\|^2 + \|\hat{u}(\cdot,t)\|^2 + \left|\hat{X}(t)\right|^2 + \hat{s}_1(t)^2 + \hat{s}_2(t)^2 + \hat{z}(t)^2 + \|v(\cdot,t)\|^2 + \|u(\cdot,t)\|^2 + |X(t)|^2 + s_1(t)^2 + s_2(t)^2 + z(t)^2. \tag{125}$$

2) The output-feedback controller (123) is bounded and exponentially convergent to zero.

Proof: 1) Rewrite (56)–(63) as

$$\dot{\hat{X}}(t) = A\hat{X}(t) + B\hat{v}(0,t) + \Gamma_0(t)\tilde{u}(l(t),t) \tag{126}$$

$$\hat{u}_t(x,t) = -p_1 \hat{u}_x(x,t) + c_1 \hat{v}(x,t) + \Gamma_1(x,t)\tilde{u}(l(t),t) \tag{127}$$

$$\hat{v}_t(x,t) = p_2 \hat{v}_x(x,t) + c_2 \hat{u}(x,t) + \Gamma_2(x,t)\tilde{u}(l(t),t) \tag{128}$$

$$\hat{u}(0,t) = q\hat{v}(0,t) + C\hat{X}(t) \tag{129}$$

$$\hat{v}(l(t),t) = \hat{s}_1(t) + \tilde{s}_1(t) \tag{130}$$

$$\dot{\hat{s}}_1(t) = c_3 \hat{s}_2(t) + f_1\left(\hat{s}_1(t), \int_0^{l(t)} \hat{u}(y,t)dy\right) + \mu_1 \tilde{s}_1(t) \tag{131}$$

$$\dot{\hat{s}}_2(t) = f_2(\hat{s}_1(t), \hat{s}_2(t), \hat{u}(l(t),t)) + \hat{z}(t) + \tilde{z}(t) + \mu_2 \tilde{s}_1(t) \tag{132}$$

$$\dot{\hat{z}}(t) = c_4 \hat{z}(t) + r\hat{u}(l(t),t) + r\tilde{u}(l(t),t) + \mu_3 \tilde{z}(t) + U_{of}(t) \tag{133}$$

which has the same structure with the original system (1)–(8) plus the injections $\tilde{u}(l(t),t)$, $\tilde{s}_1(t)$, $\tilde{z}(t)$. Applying transformations (10), (11), (18), (19), (28), (29), and (35) (note that all states in the transformation should be added a "hat," such as "$\hat{u}$"), through same steps in Section III, we can arrive the target system-$(\hat{\alpha}, \hat{\beta}, \hat{X}, \hat{y}_1, \hat{y}_2, \hat{\mathcal{E}}, \tilde{u}(l(t),t), \tilde{z}(t), \tilde{s}_1(t), \dot{\tilde{s}}_1(t))$, the main body-$(\hat{\alpha}, \hat{\beta}, \hat{X}, \hat{y}_1, \hat{y}_2, \hat{\mathcal{E}})$ of which has the same structure with the exponentially stable target system in the state-feedback design, plus several observer error injections $\tilde{u}(l(t),t), \tilde{z}(t), \tilde{s}_1(t), \dot{\tilde{s}}_1(t)$. Recalling Theorem 3 and Lemma 3, we have $\tilde{u}(l(t),t), \tilde{z}(t), \tilde{s}_1(t)$ are exponentially convergent to zero. According to (70), (F.2), we have

$$\dot{\tilde{s}}_1(t)^2 \leq 3c_3^2 \tilde{s}_2(t)^2 + 3\tilde{f}_1^2 + 3\mu_1^2 \tilde{s}_1(t)^2 \leq 3c_3^2 \tilde{s}_2(t)^2 + 3\tilde{\gamma}_1^2 \|\tilde{\alpha}(\cdot,t)\|^2 + 3(\mu_1^2 + \gamma_1^2)\tilde{s}_1(t)^2. \tag{134}$$

Thus, $\dot{\tilde{s}}_1(t)$ is also exponentially convergent to zero recalling Theorem 3.

Define a Lyapunov function as

$$V_{of} = \hat{X}(t)^T P_2 \hat{X}(t) + \frac{\bar{a}_1}{2}\int_0^{l(t)} e^{\bar{\delta}_1 x}\hat{\beta}(x,t)^2 dx + \frac{1}{2}\hat{\mathcal{E}}(t)^2 + \frac{\bar{b}_1}{2}\int_0^{l(t)} e^{-\bar{\delta}_1 x}\hat{\alpha}(x,t)^2 dx + \frac{1}{2}\hat{y}_1(t)^2 + \frac{1}{2}\hat{y}_2(t)^2 + R_{Ve}V_e(t) + R_z \tilde{z}(t)^2 \tag{135}$$

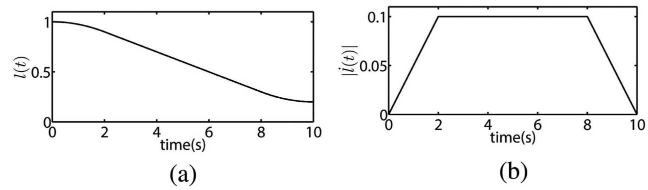

Fig. 3. Moving boundary and its velocity. (a) $l(t)$. (b) $|\dot{l}(t)|$.

where $\bar{a}_1$, $\bar{b}_1$, $\bar{\delta}_1$, R_{Ve}, R_z are positive constants, and $P_2 = P_2^T > 0$ being the solution to the Lyapunov equation $P_2(A + B\kappa) + (A + B\kappa)^T P_2 = -Q_2$ for some $Q_2 = Q_2{}^T > 0$.

Through the same steps in Theorem 1, using (134), (122), (109), we obtain $\dot{V}_{of} \leq -\lambda_{of} V_{of}(t)$ for some positive λ_{of}. We then obtain $\Omega_4(t) \leq \Upsilon_{4a}\Omega_4(0)e^{-\lambda_{of}t}$, where $\Omega_4(t) = \|\hat{\alpha}(\cdot,t)\|^2 + \|\hat{\beta}(\cdot,t)\|^2 + |\hat{X}(t)|^2 + \hat{y}_1(t)^2 + \hat{y}_2(t)^2 + \hat{\mathcal{E}}(t)^2 + |\tilde{S}(t)|^2 + |\tilde{Y}(t)|^2 + \|\tilde{\alpha}(\cdot,t)\|^2 + \|\tilde{\beta}(\cdot,t)\|^2 + \tilde{z}(t)^2$, for some positive Υ_{4a}. Applying all transformations and their inverses, through same steps with (50)–(52), we have

$$\bar{\Omega}(t) \leq \Upsilon_{4b}\bar{\Omega}(0)e^{-\lambda_{of}t} \tag{136}$$

where Υ_{4b} is a positive constant, and

$$\bar{\Omega}(t) = \|\hat{u}(\cdot,t)\|^2 + \|\hat{v}(\cdot,t)\|^2 + \left|\hat{X}(t)\right|^2 + \hat{s}_1(t)^2 + \hat{s}_2(t)^2 + \hat{z}(t)^2 + \|\tilde{u}(\cdot,t)\|^2 + \|\tilde{v}(\cdot,t)\|^2 + |\tilde{X}(t)|^2 + \tilde{s}_1(t)^2 + \tilde{s}_2(t)^2 + \tilde{z}(t)^2.$$

Then recalling (64) and applying Cauchy–Schwarz inequality, we thus obtain (124).

2) In order to prove the boundedness and exponential convergence of the output-feedback controller (123), considering the above exponential stability results in 1), the exponential convergence to zero of $\hat{u}(l(t),t)$ is required additionally. It can be obtained by the exponential stability estimate in the sense of $\|\hat{u}_x(\cdot,t)\| + \|\hat{v}_x(\cdot,t)\|$ which can be proved through same steps as Lemma 1 with recalling Lemma 3 and Theorem 3. Then through the same steps in Theorem 2, we have the output-feedback controller (123) is bounded and exponentially convergent to zero as well.

The proof of Theorem 4 is completed. ■

VI. Simulation

Consider the system given by

$$\dot{X}(t) = 0.4X(t) + v(0,t) \tag{137}$$

$$u_t(x,t) = -u_x(x,t) + 0.5v(x,t) \tag{138}$$

$$v_t(x,t) = v_x(x,t) + 0.5u(x,t) \tag{139}$$

$$u(0,t) = v(0,t) + X(t),\ v(l(t),t) = s_1(t) \tag{140}$$

$$\dot{s}_1(t) = s_2(t) + s_1(t)^2 + \int_0^{l(t)} u(x,t)dx \tag{141}$$

$$\dot{s}_2(t) = s_1(t)s_2(t) + u(l(t),t) + z(t) \tag{142}$$

$$\dot{z}(t) = 0.5z(t) + u(l(t),t) + U(t) \tag{143}$$

$x \in [0, l(t)]$. $l(t)$ is a preknown function decreasing from $l(0) = 1$ to 0.2 during 10 s, as shown in Fig 3. The initial values are given as $u(x,0) = 3\sin(4\pi x)$, $v(x,0) = 3\sin(4\pi x)$, $X(0) =$

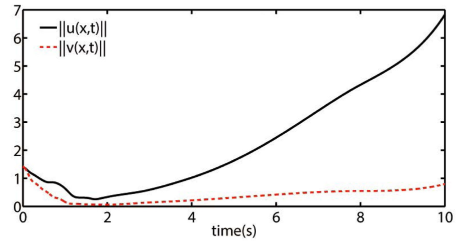

Fig. 4. Open-loop responses of $\|u(\cdot,t)\|$ and $\|v(\cdot,t)\|$.

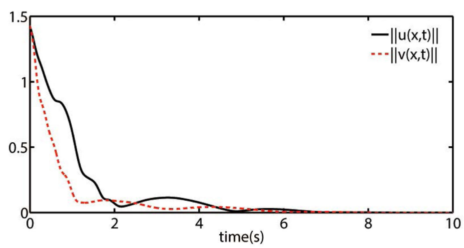

Fig. 5. Responses of $\|u(\cdot,t)\|$ and $\|v(\cdot,t)\|$ under the proposed output-feedback controller.

$u(0,0)-v(0,0)$, $s_1(0)=v(l(0),0)$, $s_2(0)=z(0)=0$. The initial values of the observer are given as $\hat{u}(x,0)=u(x,0)+0.2\sin(2\pi(l(0)-x))$, $\hat{v}(x,0)=v(x,0)+0.2\sin(2\pi(l(0)-x))$, $\hat{X}(0)=\hat{u}(0,0)-\hat{v}(0,0)$, $\hat{s}_1(0)=\hat{v}(l(0),0)$, $\hat{s}_2(0)=s_2(0)+0.5$, $\hat{Z}(0)=Z(0)+0.5$, where the additional terms are initial observer errors.

The simulation is performed by the finite-difference method for the discretization in time and space after converting the time-varying domain PDE to a fixed domain PDE via introducing $\breve{\xi}=\frac{x}{l(t)}$, and then the time step and space step are chosen as 0.001 and 0.02 respectively. Kernels (12)–(17), (87)–(90), (98), (99) used in the control input are also solved by the finite difference method. The control parameters are chosen as $c_1=80$, $c_2=150$, $\bar{a}_0=350$, $\kappa=-10$, $L_0=10$, $\mu_1=\mu_2=\mu_3=5$. The simulation results are shown following.

Comparing Fig. 4 which shows the open-loops responses of $\|u(\cdot,t)\|$, $\|v(\cdot,t)\|$ and Fig. 5 which gives the closed-loop responses of $\|u(\cdot,t)\|$, $\|v(\cdot,t)\|$, as one can observe, in the latter case convergence to zero is achieved, whereas the states grow unbounded in the former case. According to Fig. 6, we see that the responses of the ODE-$z(t)$, the nonlinear ODE-$(s_1(t),s_2(t))$ and the ODE-$X(t)$ at the opposite boundary converge to zero under the proposed output-feedback controller. Moreover, in Figs. 7 and 8, it can be observed that the proposed observer converge to the actual plant for both PDE and ODE states. Note that because $v(l(t),t)$ and $z(t)$ are measurable, $\tilde{s}_1(t)$ and $\tilde{z}(t)$ are at a small magnitude and fast convergent to zero, the curves of which are omitted here due to the space limit. In Fig. 9, it is shown that the observer-based output-feedback control input is bounded and convergent to zero.

VII. Conclusion

In this article, we address the output-feedback control problem for a particular class of coupled hyperbolic PDEs sandwiched between a cascade of an ODE and a nonlinear ODE on the actuated side and a linear ODE on the opposite side, with a PDE domain that is time-varying. First, a state-feedback control design is proposed to exponentially stabilize the overall system via a series of transformations, and then a state-observer is constructed to recover the overall system only using available boundary values at the actuated side, to build a "collocated" type observer-based output-feedback controller. The exponential stability results of the closed-loop system and the observer error system, and the boundedness and exponential convergence of the control input are proved in this article. A simulation example is conducted to verify the effectiveness of the proposed controller and observer. The proposed design in this article can be applied into brake control of a mining cable elevator.

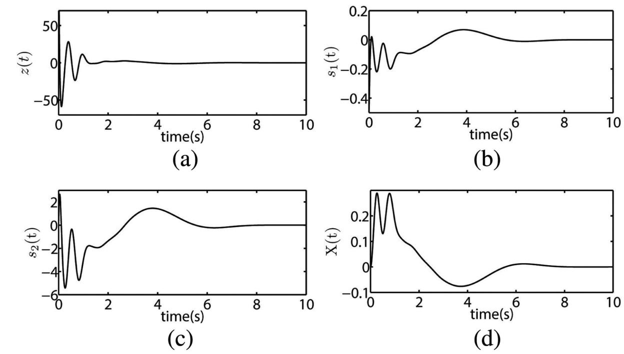

Fig. 6. Responses of ODE states $z(t), s_1(t), s_2(t), X(t)$ under the proposed output-feedback controller. (a) $z(t)$. (b) $s_1(t)$. (c) $s_2(t)$. (d) $X(t)$.

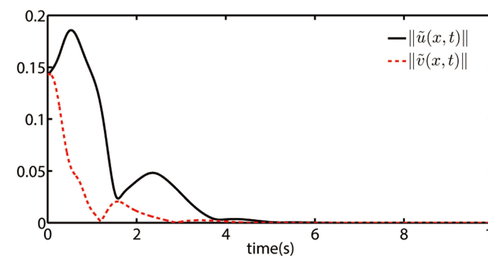

Fig. 7. Observer errors of $\|\tilde{u}(\cdot,t)\|$, $\|\tilde{v}(\cdot,t)\|$.

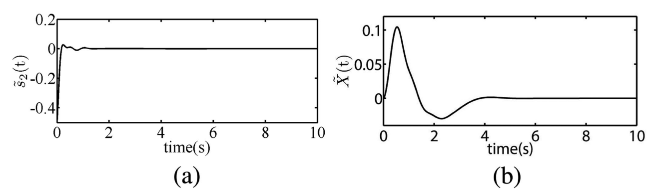

Fig. 8. Observer errors of $\tilde{s}_2(t), \tilde{X}(t)$. (a) $\tilde{s}_2(t)$. (b) $\tilde{X}(t)$.

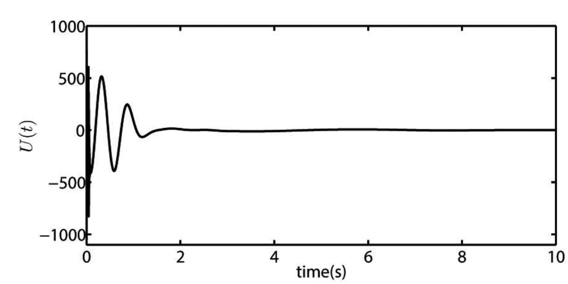

Fig. 9. Output-feedback control input.

In the future work, it is of interest to extend the control design to a more complicated and practical case with some system parameters being unknown, where an adaptive design should be developed.

APPENDIX

A. Expression of $\mathcal{F}$

$$\begin{aligned}&\mathcal{F}(\beta(l(t),\beta(0,t),\alpha(l(t),t),\alpha(0,t),\beta(x,t),\alpha(x,t),X(t))\\&=h_1(l(t))\beta(l(t),t)+h_2(l(t))\beta(0,t)\\&\quad+h_3(l(t))\alpha(l(t),t)+h_4(l(t))\alpha(0,t)\\&\quad+\int_0^{l(t)}h_5(l(t),y)\beta(y,t)dy\\&\quad+\int_0^{l(t)}h_6(l(t),y)\alpha(y,t)dy+H_7(l(t))X(t)\end{aligned}\tag{A.1}$$

where

$$h_1(l(t))=-p_2\phi(l(t),l(t))-\dot{l}(t)\phi(l(t),l(t))\tag{A.2}$$

$$h_2(l(t))=p_2\phi(l(t),0)-\gamma(l(t))B\tag{A.3}$$

$$h_3(l(t))=p_1\psi(l(t),l(t))-\dot{l}(t)\psi(l(t),l(t))\tag{A.4}$$

$$h_4(l(t))=-p_1\psi(l(t),0)\tag{A.5}$$

$$\begin{aligned}&h_5(l(t),y)=\Big(p_2\phi(l(t),l(t))+\dot{l}(t)\phi(l(t),l(t))\Big)\mathcal{M}(l(t),y)\\&+p_2\phi_y(l(t),y)-c_1\psi(l(t),y)-\dot{l}(t)\phi_x(l(t),y)\\&-\int_y^{l(t)}\Big(p_2\phi_y(l(t),z)-c_1\psi(l(t),z)\\&-\dot{l}(t)\phi_x(l(t),z)\Big)\mathcal{M}(z,y)dz\end{aligned}\tag{A.6}$$

$$\begin{aligned}&h_6(l(t),y)=p_1\psi_y(l(t),y)+c_2\phi(l(t),y)+\dot{l}(t)\psi_x(l(t),y)\\&-\Big(p_2\phi(l(t),l(t))+\dot{l}(t)\phi(l(t),l(t))\Big)\mathcal{D}(l(t),y)\\&-\int_y^{l(t)}\Big(p_2\phi_y(l(t),z)-c_1\psi(l(t),z)\\&-\dot{l}(t)\phi_x(l(t),z)\Big)\mathcal{D}(z,y)dz\end{aligned}\tag{A.7}$$

$$\begin{aligned}&H_7(l(t))=\Big(p_2\phi(l(t),l(t))+\dot{l}(t)\phi(l(t),l(t))\Big)\mathcal{J}(l(t))\\&-\gamma(l(t))A-\dot{l}(t)\gamma'(l(t))-(p_2\phi(l(t),0)-\gamma(l(t))B)\,\mathcal{J}(0)\\&-\int_0^{l(t)}\Big(p_2\phi_y(l(t),y)-c_1\psi(l(t),y)\\&-\dot{l}(t)\phi_x(l(t),y)\Big)\mathcal{J}(y)dy.\end{aligned}\tag{A.8}$$

B. Calculation of $\dot{V}_1(t)$

Taking the time derivative of (43) along (20)–(23), we obtain

$$\begin{aligned}&\dot{V}_1(t)\le-\lambda_{\min}(Q_1)|X(t)|^2+2X^TP_1B\beta(0,t)\\&+\frac{p_2}{2}a_1e^{\delta_1 l(t)}\beta(l(t),t)^2-\frac{p_2}{2}a_1\beta(0,t)^2\\&+\frac{a_1\dot{l}(t)}{2}e^{\delta_1 l(t)}\beta(l(t),t)^2+\frac{b_1\dot{l}(t)}{2}e^{-\delta_1 l(t)}\alpha(l(t),t)^2\\&-\frac{p_2}{2}\delta_1a_1\int_0^{l(t)}e^{\delta_1 x}\beta(x,t)^2dx\\&-\frac{p_1}{2}\delta_1b_1\int_0^{l(t)}e^{-\delta_1 x}\alpha(x,t)^2dx\\&-\frac{p_1}{2}b_1e^{-\delta_1 l(t)}\alpha(l(t),t)^2+\frac{p_1}{2}b_1\alpha(0,t)^2\\&+b_1c_1\int_0^{l(t)}e^{-\delta_1 x}\alpha(x,t)\Big(\beta(x,t)-\int_0^x\mathcal{D}(x,y)\alpha(y,t)dy\\&-\int_0^x\mathcal{M}(x,y)\beta(y,t)dy-\mathcal{J}(x)X(t)\Big)dx.\end{aligned}\tag{B.1}$$

Recalling (23), using Young's inequality and Cauchy–Schwarz inequality for the last part in (B.1) yields the existence of $\xi>0$ such that

$$\begin{aligned}&\dot{V}_1(t)\le-\left(\frac{1}{2}\lambda_{\min}(Q_1)-p_1b_1|C_0|^2\right)|X(t)|^2\\&-\left(\frac{p_2}{2}a_1-p_1b_1q^2-\frac{4|P_1B|}{\lambda_{\min}(Q_1)}\right)\beta(0,t)^2\\&-\left(\frac{p_2}{2}\delta_1a_1-b_1\xi-b_1\frac{\xi}{\delta_1}\right)\int_0^{l(t)}\beta(x,t)^2dx\\&-\left(\frac{p_2\delta_1b_1}{2}-\frac{b_1\xi}{\delta_1}-\frac{\xi b_1^2}{\lambda_{\min}(Q_1)}-b_1\xi\right)e^{-\delta_1 L}\int_0^{l(t)}\alpha(x,t)^2dx\\&-(p_1-\dot{l}(t))\frac{b_1}{2}e^{-\delta_1 l(t)}\alpha(l(t),t)^2\\&+(p_2+\dot{l}(t))\frac{a_1}{2}e^{\delta_1 l(t)}\beta(l(t),t)^2\end{aligned}\tag{B.2}$$

where

$$\xi=\max\left\{\frac{c_1}{4}[\bar{\mathcal{D}}(1+L)+\bar{\mathcal{M}}],L\bar{\mathcal{J}}^2c_1^2,\frac{1}{2}c_1,\frac{c_1}{4}\bar{\mathcal{D}}L\right\}\tag{B.3}$$

and

$$\bar{\mathcal{D}}=\max_{0\le y\le x\le L}\{|\mathcal{D}(x,y)|\}\tag{B.4}$$

$$\bar{\mathcal{M}}=\max_{0\le y\le x\le L}\{|\mathcal{M}(x,y)|\}\tag{B.5}$$

$$\bar{\mathcal{J}}=\max_{0\le x\le L}\{|\mathcal{J}(x)|\}.\tag{B.6}$$

Choose parameters b_1,δ_1,a_1 to satisfy

$$0<b_1<\frac{\lambda_{\min}(Q_1)}{2p_1|C_0|^2}\tag{B.7}$$

$$\delta_1>\max\left\{1,\frac{2}{p_2}\left(2\xi+\frac{\xi b_1}{\lambda_{\min}(Q_1)}\right)\right\}\tag{B.8}$$

$$a_1>\max\left\{\frac{8|P_1B|}{p_2\lambda_{\min}(Q_1)}+2q^2b_1\frac{p_1}{p_2},\ \frac{2b_1\xi}{p_2\delta_1}+\frac{2b_1\xi}{p_2\delta_1^2}\right\}\tag{B.9}$$

we arrive at (44), where

$$\eta_1=\frac{1}{2}\lambda_{\min}(Q_1)-p_1b_1|C_0|^2>0\tag{B.10}$$

$$\eta_2=\frac{p_2}{2}a_1-p_1b_1q^2-\frac{4|P_1B|}{\lambda_{\min}(Q_1)}>0\tag{B.11}$$

$$\eta_3=\frac{p_2}{2}\delta_1a_1-b_1\xi-b_1\frac{\xi}{\delta_1}>0\tag{B.12}$$

$$\eta_4 = \left(\frac{p_2\delta_1 b_1}{2} - \frac{b_1\xi}{\delta_1} - \frac{\xi b_1^2}{\lambda_{\min}(Q_1)} - b_1\xi\right)e^{-\delta_1 L} > 0. \tag{B.13}$$

C. Calculation of $\dot V(t)$

Taking the time derivative of (45) and recalling (39) (36), and (44) with (A.1)–(A.8), we have

$$\begin{aligned}\dot V \le & -\eta_1|X(t)|^2 - \eta_2\beta(0,t)^2 - \eta_3\int_0^{l(t)}\beta(x,t)^2dx \\ & -\eta_4\int_0^{l(t)}\alpha(x,t)^2dx - \eta_5\alpha(l(t),t)^2 + \eta_6\beta(l(t),t)^2 \\ & -\bar c_1 y_1(t)^2 - \bar c_2 y_2(t)^2 + y_1(t)f_1 \\ & + y_1(t)\Big(h_1(l(t))\beta(l(t),t) + h_2(l(t))\beta(0,t) \\ & + h_3(l(t))\alpha(l(t),t) + h_4(l(t))\alpha(0,t) \\ & + \int_0^{l(t)} h_5(l(t),y)\beta(y,t)dy + \int_0^{l(t)} h_6(l(t),y)\alpha(y,t)dy \\ & + H_7(l(t))X(t)\Big) + y_2(t)\mathcal{E}(t) + y_2(t)f_2 + \frac{\bar c_1}{c_3}y_2(t)\dot y_1(t) \\ & - k_{\mathcal{E}}\mathcal{E}(t)^2 + \mathcal{E}(t)\bar c_2\dot y_2(t) + \mathcal{E}(t)c_3\dot y_1(t).\end{aligned} \tag{C.1}$$

Recalling Assumptions 1 and 2 and (28), (29), (31), we have

$$\begin{aligned}f_1^2 \le & \gamma_{f1}\left(4y_1(t)^2 + 4\|\beta(\cdot,t)\|^2\right. \\ & \left.+5\|\alpha(\cdot,t)\|^2 + 4|X(t)|^2\right)\end{aligned} \tag{C.2}$$

$$\begin{aligned}f_2^2 \le \gamma_{f2}\Bigg(&\left(4+\frac{2\bar c_1^2}{c_3^2}\right)y_1(t)^2 + 4\|\beta(\cdot,t)\|^2 + 4\|\alpha(\cdot,t)\|^2 \\ & + 4|X(t)|^2 + 2y_2(t)^2 + \alpha(l(t),t)^2\Bigg)\end{aligned} \tag{C.3}$$

where γ_{f1}, γ_{f2} are positive constants depending on kernels $\mathcal{D}, \mathcal{M}, \mathcal{J}$. The omitted arguments of f_1, f_2 are same as those in (25) and (26).

Applying Young's inequality, Cauchy–Schwarz inequality into the ninth and tenth terms, and $y_2(t)\mathcal{E}(t) + y_2(t)f_2 + \frac{\bar c_1}{c_3}y_2(t)\dot y_1(t) + \mathcal{E}(t)\bar c_2\dot y_2(t) + \mathcal{E}(t)c_3\dot y_1(t)$ in (C.1), where (26), (29), (31), (35) are used to rewrite $\dot y_2(t)$ as $\dot y_2(t) = f_2 + \mathcal{E}(t) - \bar c_2 y_2(t) - c_3 y_1(t) + \frac{\bar c_1}{c_3}\dot y_1(t)$, using (C.2)–(C.3) to replace the resulting f_1^2, f_2^2, recalling (23), (28) to rewrite the resulting $\alpha(0,t)^2$, $\beta(l(t),t)^2$, respectively, we have

$$\begin{aligned}\dot V \le & -\Big(\eta_1 - r_7\left|\bar H_7\right|^2 - 2r_4\bar h_4^2\left|C_0\right|^2 - 4r_8\gamma_{f1} - 4r_9\gamma_{f2} \\ & - 4r_{11}\gamma_{f2}\Big)|X(t)|^2 - \left(\eta_2 - \bar h_2^2 r_2 - 2r_4\bar h_4^2 q^2\right)\beta(0,t)^2 \\ & - \Big(\eta_3 - r_5\, h_{5\max}^2 L - 4r_8\gamma_{f1} - 4r_9\gamma_{f2} \\ & - 4r_{11}\gamma_{f2}\Big)\int_0^{l(t)}\beta(x,t)^2dx - \Big(\eta_4 - r_6\, h_{6\max}^2 L \\ & - 5r_8\gamma_{f1} - 4r_9\gamma_{f2} - 4r_{11}\gamma_{f2}\Big)\int_0^{l(t)}\alpha(x,t)^2dx \\ & - \left(\eta_5 - \bar h_3^2 r_3 - r_9\gamma_{f2} - r_{11}\gamma_{f2}\right)\alpha(l(t),t)^2 \\ & - \Big(\bar c_1 - 1 - \frac{1}{4r_2} - \frac{1}{4r_3} - \frac{1}{4r_4} - \frac{1}{4r_5} - \frac{1}{4r_6} - \frac{1}{4r_7} - \eta_6 \\ & - \frac{1}{4r_8} - 4r_8\gamma_{f1} - \bar h_1 - \left(4 + \frac{2\bar c_1^2}{c_3^2}\right)r_9\gamma_{f2} \\ & - \left(4 + \frac{2\bar c_1^2}{c_3^2}\right)r_{11}\gamma_{f2}\Big)y_1(t)^2 \\ & - \left(\bar c_2 - \frac{1}{4r_9} - \frac{\bar c_1^2}{4c_3^2 r_{10}} - \frac{3}{2} - 2r_{11}\gamma_{f2} - 2r_9\gamma_{f2}\right)y_2(t)^2 \\ & - \left((k_{\mathcal{E}} - \bar c_2) - \frac{1}{2} - \frac{\bar c_2^4}{4} - \frac{\bar c_2^2}{4r_{11}} - \frac{\bar c_2^2\bar c_1^2}{4r_{12}c_3^2} - \frac{\bar c_2^2 c_3^2}{4} - \frac{c_3^2}{4r_{13}}\right)\mathcal{E}(t)^2 \\ & + (r_{10} + r_{12} + r_{13})\dot y_1(t)^2\end{aligned} \tag{C.4}$$

where $r_1, \ldots, r_{13}$ are positive constants from using Young's inequality,

$$h_{5\max} = \max_{x\in[0,L], l(t)\in[0,L]}\{|h_5(x,l(t))|\} \tag{C.5}$$

$$h_{6\max} = \max_{x\in[0,L], l(t)\in[0,L]}\{|h_6(x,l(t))|\} \tag{C.6}$$

and $\bar h_1$, $\bar h_2$, $\bar h_3$, $\bar h_4$, $\bar H_7$ are maximum values of $|h_1(l(t))|$, $|h_2(l(t))|$, $|h_3(l(t))|$, $|h_4(l(t))|$, $|H_7(l(t))|$ for $l(t)\in[0,L]$ in (A.2)–(A.8).

According to (25), (28), (29), and (31) with (A.1)–(A.8), we have

$$\begin{aligned}\dot y_1(t)^2 \le \bar\xi_c\Big(&\bar c_1 y_1(t)^2 + y_1(t)^2 + y_2(t)^2 + f_1^2 + \beta(0,t)^2 \\ & + \alpha(l(t),t)^2 + \alpha(0,t)^2 + \|\beta(\cdot,t)\|^2 \\ & + \|\alpha(\cdot,t)\|^2 + X(t)^2\Big)\end{aligned} \tag{C.7}$$

for some positive constants $\bar\xi_c$ depending on kernels $\mathcal{D}, \mathcal{M}, \mathcal{J}$ and gains (A.2)–(A.8).

Inserting (C.7) into (C.4) to replace $\dot y_1(t)^2$ with using (C.2), we arrive at

$$\begin{aligned}\dot V \le & -\Big(\eta_1 - r_7\left|\bar H_7\right|^2 - 2r_4\bar h_4^2\left|C_0\right|^2 - 4r_8\gamma_{f1} - 4r_9\gamma_{f2} \\ & - 4r_{11}\gamma_{f2} - (r_{10}+r_{12}+r_{13})\bar\xi_c - 2(r_{10}+r_{12}+r_{13})\bar\xi_c|C_0|^2 \\ & - 4\gamma_{f_1}(r_{10}+r_{12}+r_{13})\bar\xi_c\Big)|X(t)|^2 - \Big(\eta_2 - \bar h_2^2 r_2 - 2r_4\bar h_4^2 q^2 \\ & - (1+2q^2)(r_{10}+r_{12}+r_{13})\bar\xi_c\Big)\beta(0,t)^2 - \Big(\eta_3 - r_5\, h_{5\max}^2 L \\ & - 4r_8\gamma_{f1} - 4r_9\gamma_{f2} - 4r_{11}\gamma_{f2} - (r_{10}+r_{12}+r_{13})\bar\xi_c \\ & - 4\gamma_{f_1}(r_{10}+r_{12}+r_{13})\bar\xi_c\Big)\int_0^{l(t)}\beta(x,t)^2dx - \Big(\eta_4 - 5r_8\gamma_{f1} \\ & - r_6\, h_{6\max}^2 L - 4r_9\gamma_{f2} - 4r_{11}\gamma_{f2} - (r_{10}+r_{12}+r_{13})\bar\xi_c\end{aligned}$$

$$- 5\gamma_{f_1}(r_{10}+r_{12}+r_{13})\bar{\xi}_c\Big)\int_0^{l(t)}\alpha(x,t)^2dx - \big(\eta_5 - \bar{h}_3^2 r_3 - r_9\gamma_{f2} - r_{11}\gamma_{f2} - (r_{10}+r_{12}+r_{13})\bar{\xi}_c\big)\,\alpha(l(t),t)^2$$
$$- \Bigg(\bar{c}_1 - 1 - \frac{1}{4r_2} - \frac{1}{4r_3} - \frac{1}{4r_4} - \frac{1}{4r_5} - \frac{1}{4r_6} - \frac{1}{4r_7} - \eta_6 - \frac{1}{4r_8} - 4r_8\gamma_{f1} - \bar{h}_1 - \left(4+\frac{2\bar{c}_1^2}{c_3^2}\right)(r_9+r_{11})\gamma_{f2} - (1+4\gamma_{f_1}+\bar{c}_1)(r_{10}+r_{12}+r_{13})\bar{\xi}_c\Bigg)y_1(t)^2$$
$$- \left(\bar{c}_2 - \frac{1}{4r_9} - \frac{\bar{c}_1^2}{4c_3^2 r_{10}} - \frac{3}{2} - 2r_{11}\gamma_{f2} - 2r_9\gamma_{f2} - (r_{10}+r_{12}+r_{13})\bar{\xi}_c\right)y_2(t)^2 - \left(\bar{a}_0 - c_4 - \bar{c}_2 - \frac{1}{2} - \frac{\bar{c}_2^4}{4} - \frac{\bar{c}_2^2}{4r_{11}} - \frac{\bar{c}_1^2\bar{c}_2^2}{4r_{12}c_3^2} - \frac{\bar{c}_2^2c_3^2}{4} - \frac{c_3^2}{4r_{13}}\right)\mathcal{E}(t)^2 \tag{C.8}$$

where $k_{\mathcal{E}} = \bar{a}_0 - c_4$ is recalled. Choosing small enough positive $r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}$ and making the control parameters $\bar{c}_1, \bar{c}_2, \bar{a}_0$ to satisfy

$$\bar{c}_1 > 1 + \frac{1}{4r_2} + \frac{1}{4r_3} + \frac{1}{4r_4} + \frac{1}{4r_5} + \frac{1}{4r_6} + \frac{1}{4r_7} + \eta_6 + \frac{1}{4r_8} + 4r_8\gamma_{f1} + \bar{h}_1 \tag{C.9}$$

$$\bar{c}_2 > \frac{1}{4r_9} + \frac{\bar{c}_1^2}{4c_3^2 r_{10}} + \frac{3}{2} + 2r_{11}\gamma_{f2} + 2r_9\gamma_{f2} + (r_{10}+r_{12}+r_{13})\bar{\xi}_c, \tag{C.10}$$

$$\bar{a}_0 > c_4 + \bar{c}_2 + \frac{1}{2} + \frac{\bar{c}_2^4}{4} + \frac{\bar{c}_2^2}{4r_{11}} + \frac{\bar{c}_1^2\bar{c}_2^2}{4r_{12}c_3^2} + \frac{\bar{c}_2^2c_3^2}{4} + \frac{c_3^2}{4r_{13}} \tag{C.11}$$

we obtain (48) with

$$\hat{\eta}_0 = \eta_2 - \bar{h}_2^2 r_2 - 2r_4\bar{h}_4^2 q^2 - (1+2q^2)(r_{10}+r_{12}+r_{13})\bar{\xi}_c > 0 \tag{C.12}$$

$$\hat{\eta}_1 = \eta_5 - \bar{h}_3^2 r_3 - r_9\gamma_{f2} - r_{11}\gamma_{f2} - (r_{10}+r_{12}+r_{13})\bar{\xi}_c > 0. \tag{C.13}$$

D. Proof of Lemma 1

Differentiating (21) and (22) with respect to x, differentiating (23) with respect to t, we have

$$\alpha_{xt}(x,t) = -p_1\alpha_{xx}(x,t) + c_1\beta_x(x,t) - c_1\mathcal{J}'(x)X(t) - c_1\mathcal{D}(x,x)\alpha(x,t) - c_1\mathcal{M}(x,x)\beta(x,t) - c_1\int_0^x \mathcal{D}_x(x,y)\alpha(y,t)dy - c_1\int_0^x \mathcal{M}_x(x,y)\beta(y,t)dy \tag{D.1}$$

$$\beta_{xt}(x,t) = p_2\beta_{xx}(x,t) \tag{D.2}$$

$$\alpha_x(0,t) = -q\frac{p_2}{p_1}\beta_x(0,t) - \frac{1}{p_1}\left(C_0(A+B\kappa) - c_1\mathcal{J}(0)\right)X(t) - \frac{1}{p_1}(C_0B - c_1)\beta(0,t). \tag{D.3}$$

Defining

$$\bar{A} = \frac{b_2}{2}\int_0^{l(t)} e^{-\delta_2 x}\alpha_x(x,t)^2dx + \frac{a_2}{2}\int_0^{l(t)} e^{\delta_2 x}\beta_x(x,t)^2dx \tag{D.4}$$

where b_2 is an arbitrary positive constant which can adjust the convergence rate and the positive constants δ_2, a_2 will be chosen later.

Taking the derivative of (D.4) along (D.1), (D.2), we obtain

$$\begin{aligned}\dot{\bar{A}} = & -\frac{p_1}{2}b_2 e^{-\delta_2 l(t)}\alpha_x(l(t),t)^2 + \frac{p_1}{2}b_2\alpha_x(0,t)^2 + \frac{b_2\dot{l}(t)}{2}e^{-\delta_2 l(t)}\alpha_x(l(t),t)^2 - \frac{p_1}{2}b_2\delta_2\int_0^{l(t)} e^{-\delta_2 x}\alpha_x(x,t)^2dx \\ & + (p_2+\dot{l}(t))\frac{a_2}{2}e^{\delta_2 l(t)}\beta_x(l(t),t)^2 - \frac{p_2}{2}a_2\beta_x(0,t)^2 - \frac{p_2}{2}a_2\delta_2\int_0^{l(t)} e^{\delta_2 x}\beta_x(x,t)^2dx \\ & - \int_0^{l(t)} b_2 e^{-\delta_2 x}\alpha_x(x,t)c_1\mathcal{D}(x,x)\alpha(x,t)dx - \int_0^{l(t)} b_2 e^{-\delta_2 x}\alpha_x(x,t)c_1\mathcal{M}(x,x)\beta(x,t)dx \\ & - \int_0^{l(t)} b_2 e^{-\delta_2 x}\alpha_x(x,t)c_1\int_0^x \mathcal{D}_x(x,y)\alpha(y,t)dydx - \int_0^{l(t)} b_2 e^{-\delta_2 x}\alpha_x(x,t)c_1\int_0^x \mathcal{M}_x(x,y)\beta(y,t)dydx \\ & + \int_0^{l(t)} b_2 e^{-\delta_2 x}\alpha_x(x,t)c_1\beta_x(x,t)dx - \int_0^{l(t)} b_2 e^{-\delta_2 x}\alpha_x(x,t)c_1\mathcal{J}'(x)X(t)dx.\end{aligned} \tag{D.5}$$

Using Young's inequality and Cauchy–Schwarz inequality into the last six terms in (D.5) yields the existence of $\xi_2 > 0$ such that

$$\begin{aligned}\dot{\bar{A}}(t) \le & -(p_1 - \dot{l}(t))\frac{b_2}{2}e^{-\delta_2 l(t)}\alpha_x(l(t),t)^2 + \frac{p_1}{2}b_2\alpha_x(0,t)^2 + (p_2+\dot{l}(t))\frac{a_2}{2}e^{\delta_2 l(t)}\beta_x(l(t),t)^2 - \frac{p_2}{2}a_2\beta_x(0,t)^2 \\ & - \left(\frac{p_1}{2}b_2\delta_2 - 4\xi_2 b_2 - \frac{2\xi_2 b_2}{\delta_2}\right)\int_0^{l(t)} e^{-\delta_2 x}\alpha_x(x,t)^2dx - \left(\frac{p_2}{2}a_2\delta_2 - \xi_2 b_2\right)\int_0^{l(t)} e^{\delta_2 x}\beta_x(x,t)^2dx \\ & + \left(\xi_2 b_2 + \frac{\xi_2 b_2}{\delta_2}\right)\int_0^{l(t)} e^{-\delta_2 x}\alpha(x,t)^2dx + \xi_2 b_2|X(t)|^2 + \left(\xi_2 b_2 + \frac{\xi_2 b_2}{\delta_2}\right)\int_0^{l(t)} e^{\delta_2 x}\beta(x,t)^2dx.\end{aligned} \tag{D.6}$$

Note that $\alpha_x(0,t)^2$ in (D.6) can be replaced by

$$\alpha_x(0,t)^2 \le 3\frac{p_2^2}{p_1^2}q^2\beta_x(0,t)^2 + \frac{3}{p_1^2}(C_0B-c_1)^2\beta(0,t)^2 + \frac{3}{p_1^2}\left|C_0(A+B\kappa)+c_1\mathcal{J}(0)\right|^2|X(t)|^2 \quad \text{(D.7)}$$

using Cauchy–Schwarz inequality into (D.3). Recalling (22), (25) with (A.1)–(A.8), (28), (29), (31), (C.2), using Cauchy–Schwarz inequality, the positive term $(p_2+\dot{l}(t))\frac{a_2}{2}e^{\delta_2 l(t)}\beta_x(l(t),t)^2$ in (D.6) can be replaced as

$$\begin{aligned}&(p_2+\dot{l}(t))\frac{a_2}{2}e^{\delta_2 l(t)}\beta_x(l(t),t)^2\\ &\le \bar{\xi}_2 y_2(t)^2 + \bar{\xi}_3 y_1(t)^2 + \bar{\xi}_4\beta(0,t) + \bar{\xi}_5\alpha(l(t),t)^2 + \bar{\xi}_6\alpha(0,t)^2\\ &+\bar{\xi}_7\|\alpha(\cdot,t)\|^2 + \bar{\xi}_8\|\beta(\cdot,t)\|^2 + \bar{\xi}_9|X(t)|^2 \qquad \text{(D.8)}\end{aligned}$$

for some positive $\bar{\xi}_i, i=2,\ldots,9$.

We propose a Lyapunov function

$$V_2(t) = \bar{A}(t) + R_1V(t). \quad \text{(D.9)}$$

Define the norm as

$$\begin{aligned}\Omega_2(t) &= \|\beta_x(\cdot,t)\|^2 + \|\alpha_x(\cdot,t)\|^2 + \|\beta(\cdot,t)\|^2\\ &+\|\alpha(\cdot,t)\|^2 + |X(t)|^2 + y_1(t)^2 + y_2(t)^2 + \mathcal{E}(t)^2. \quad \text{(D.10)}\end{aligned}$$

We have

$$\theta_{2a}\Omega_2(t) \le V_2(t) \le \theta_{2b}\Omega_2(t) \quad \text{(D.11)}$$

for some positive θ_{2a} and θ_{2b}.

Taking the derivative of (D.9) and recalling (D.6)–(D.8), (48), we then get

$$\begin{aligned}&\dot{V}_2(t) = \dot{\bar{A}} + R_1\dot{V}\\ &\le -\left(p_1-\dot{l}(t)\right)\frac{b_2}{2}e^{-\delta_2 l(t)}\alpha_x(l(t),t)^2\\ &-\left(\frac{p_2}{2}a_2 - \frac{3p_2^2q^2}{2b_2p_1}\right)\beta_x(0,t)^2\\ &-\left(\frac{p_1}{2}b_2\delta_2 - 4\xi_2b_2 - \frac{2\xi_2b_2}{\delta_2}\right)\int_0^{l(t)} e^{-\delta_2 x}\alpha_x(x,t)^2dx\\ &-\left(\frac{p_2}{2}a_2\delta_2 - \xi_2b_2\right)\int_0^{l(t)} e^{\delta_2 x}\beta_x(x,t)^2dx - \frac{R_1}{2}\lambda V(t)\\ &-\left(R_1\hat{\eta}_0 - \frac{3b_2}{2p_1}(C_0B-c_1)^2 - \bar{\xi}_4 - 2\bar{\xi}_6q^2\right)\beta(0,t)^2\\ &-\left(\frac{R_1}{2}\theta_{1a}\lambda - \bar{\xi}_2\right)y_2(t)^2 - \left(\frac{R_1}{2}\theta_{1a}\lambda - \bar{\xi}_3\right)y_1(t)^2\\ &-\left(\frac{R_1}{2}\theta_{1a}\lambda - \xi_2b_2 - \frac{\xi_2b_2}{\delta_2} - \bar{\xi}_7\right)\int_0^{l(t)}\alpha(x,t)^2dx\\ &-\left(\frac{R_1}{2}\theta_{1a}\lambda - \xi_2b_2e^{\delta_2 L} - \frac{\xi_2b_2}{\delta_2}e^{\delta_2 L} - \bar{\xi}_8\right)\int_0^{l(t)}\beta(x,t)^2dx\\ &-\left(\frac{R_1}{2}\theta_{1a}\lambda - \xi_2b_2 - \frac{3b_2}{2p_1}|c_1\mathcal{J}(0)+C_0(A+B\kappa)|^2\right.\\ &\left.-\bar{\xi}_9 - 2\bar{\xi}_6|C_0|^2\right)|X(t)|^2 - (R_1\hat{\eta}_1-\bar{\xi}_5)\alpha(l(t),t)^2\end{aligned}$$

$$\begin{aligned}&\le -\sigma_1V_2(t) - \left(R_1\hat{\eta}_1 - \bar{\xi}_5\right)\alpha(l(t),t)^2\\ &-\left(p_1-\dot{l}(t)\right)\frac{b_2}{2}e^{-\delta_2 l(t)}\alpha_x(l(t),t)^2\\ &-\hat{\eta}_2\beta(0,t)^2 - \hat{\eta}_3\beta_x(0,t)^2 \qquad \text{(D.12)}\end{aligned}$$

for some positive σ_1, by choosing

$$\delta_2 > \max\left\{1, \frac{12\xi_2}{p_1}\right\},\quad a_2 > \max\left\{\frac{2\xi_2b_2}{p_2\delta_2}, \frac{3p_2q^2}{b_2p_1}\right\} \quad \text{(D.13)}$$

and sufficiently large R_1. Note $\hat{\eta}_2 = R_1\hat{\eta}_0 - \frac{3b_2}{2p_1}(C_0B-c_1)^2 - \bar{\xi}_4 - 2\bar{\xi}_6q^2 > 0$, $\hat{\eta}_3 = \frac{p}{2}a_2 - \frac{3p_2^2q^2}{2b_2p_1} > 0$ and $R_1\hat{\eta}_1 - \bar{\xi}_5 > 0$ according to the above choices of R_1, a_2, and $p_1 - \dot{l}(t) > 0$ by recalling Assumption 4. We thus have $\dot{V}_2(t) \le -\sigma_1V_2(t)$. It then follows that $V_2(t) \le V_2(0)e^{-\sigma_1 t}$. Recalling (D.11), we obtain

$$\Omega_2(t) \le \frac{\theta_{2b}}{\theta_{2a}}\Omega_2(0)e^{-\sigma_1 t}. \quad \text{(D.14)}$$

Differentiating (18), (19) with respect to x, we have

$$u_x(x,t) = \alpha_x(x,t) \quad \text{(D.15)}$$

$$\begin{aligned}v_x(x,t) &= \beta_x(x,t) - \int_0^x \mathcal{D}_x(x,y)\alpha(y,t)dy\\ &-\int_0^x \mathcal{M}_x(x,y)\beta(y,t)dy - \mathcal{J}'(x)X(t)\\ &-\mathcal{D}(x,x)\alpha(x,t) - \mathcal{M}(x,x)\beta(x,t). \qquad \text{(D.16)}\end{aligned}$$

Similarly differentiating (10), (11) with respect to x, together with (D.15), (D.16), using (10), (11), (18), (19), (28), (29), and (35), we have $\bar{\theta}_{2a}\Xi_1(t) \le \Omega_2(t) \le \bar{\theta}_{2b}\Xi_1(t)$ for some positive $\bar{\theta}_{2a}$, $\bar{\theta}_{2b}$, where $\Xi_1(t)$ is defined as $\Xi_1(t) = \Xi(t) + \|u_x(\cdot,t)\|^2 + \|v_x(\cdot,t)\|^2$. Therefore, one obtain

$$\|u_x(\cdot,t)\|^2 + \|v_x(\cdot,t)\|^2 \le \Xi_1(t) \le \frac{\bar{\theta}_{2b}\theta_{2b}}{\bar{\theta}_{2a}\theta_{2a}}\Xi_1(0)e^{-\sigma_1 t}.$$

Thus (54) is obtained with

$$\Upsilon_{1a} = \frac{\bar{\theta}_{2b}\theta_{2b}}{\bar{\theta}_{2a}\theta_{2a}},\ \lambda_{1a} = \sigma_1. \quad \text{(D.17)}$$

The proof of Lemma 1 is completed.

E. Proof of Theorem 2

Applying Cauchy–Schwarz inequality into (40), we have

$$\begin{aligned}|U(t)|^2 \le \xi_d\Big(&|s_1(t)|^2 + |s_2(t)|^2 + |z(t)|^2 + |X(t)|^2\\ &+|u(l(t),t)|^2 + \|u(\cdot,t)\|^2 + \|v(\cdot,t)\|^2\Big) \qquad \text{(E.1)}\end{aligned}$$

for some positive ξ_d.

Applying Cauchy–Schwarz inequality and recalling (4), (5), we have

$$\begin{aligned}|u(l(t),t)| &\le |u(0,t)| + \sqrt{L}\|u_x(\cdot,t)\|\\ &\le |qv(0,t)| + |CX(t)| + \sqrt{L}\|u_x(\cdot,t)\|\\ &\le |q||s_1(t)| + |q|\sqrt{L}\|v_x(\cdot,t)\|\\ &+|C||X(t)| + \sqrt{L}\|u_x(\cdot,t)\|. \qquad \text{(E.2)}\end{aligned}$$

Considering (E.1), (E.2), using Theorem 1 and Lemma 1, we have the control input (40) is bounded by

$$|U(t)| \leq \Upsilon_2 \left(\Xi(0) + \|u_x(\cdot,0)\|^2 + \|v_x(\cdot,0)\|^2\right)^{\frac{1}{2}} e^{-\lambda_2 t}. \tag{E.3}$$

Then (55) is obtained by recalling (50). The proof of Theorem 2 is completed.

F. Calculation of $\dot{V}_s(t)$

Taking the derivative of (118) along (108), we have

$$\dot{V}_s(t) \leq \tilde{S}(t)^T \left((A_s - \mathcal{B}C_2)^T P_0 + P_0(A_s - \mathcal{B}C_2)\right)\tilde{S}(t) + 2\tilde{S}(t)^T P_0 \tilde{f} \tag{F.1}$$

where $\tilde{f} = [\tilde{f}_1, \tilde{f}_2]^T$. Recalling (73), (74), according to Assumption 2, we have

$$\begin{aligned}\tilde{f}_1^2 &= \left| f_1\left(s_1, \int_0^{l(t)} u(x,t)dx\right) - f_1\left(\hat{s}_1, \int_0^{l(t)} \hat{u}(x,t)dx\right)\right|^2 \\ &\leq \gamma_1^2|s_1 - \hat{s}_1|^2 + \gamma_1^2\left|\int_0^{l(t)}(u(x,t) - \hat{u}(x,t))dx\right|^2 \\ &\leq \gamma_1^2\tilde{s}_1^2 + \hat{\gamma}_1^2\|\tilde{\alpha}(\cdot,t)\|^2 \end{aligned} \tag{F.2}$$

$$\begin{aligned}\tilde{f}_2^2 &= |f_2(s_1, s_2, u(l(t),t)) - f_2(\hat{s}_1, \hat{s}_2, \hat{u}(l(t),t))|^2 \\ &\leq \gamma_2^2|(s_1, s_2, u(l(t),t)) - (\hat{s}_1, \hat{s}_2, \hat{u}(l(t),t))|^2 \\ &\leq \gamma_2^2\tilde{s}_1^2 + \gamma_2^2\tilde{s}_2^2 + \gamma_2^2\tilde{\alpha}(l(t),t)^2 \end{aligned} \tag{F.3}$$

where $\hat{\gamma}_1$ is a positive constant and (78) is used. Then

$$\begin{aligned}\tilde{f}^2 &= \tilde{f}_1^2 + \tilde{f}_2^2 \\ &\leq (\gamma_1^2 + \gamma_2^2)\tilde{s}_1(t)^2 + \gamma_2^2\tilde{s}_2(t)^2 + \hat{\gamma}_1^2\|\tilde{\alpha}(\cdot,t)\|^2 + \gamma_2^2\tilde{\alpha}(l(t),t)^2 \\ &\leq (\gamma_1^2 + 2\gamma_2^2)\left|\tilde{S}(t)\right|^2 + \hat{\gamma}_1^2\|\tilde{\alpha}(\cdot,t)\|^2 + \gamma_2^2\tilde{\alpha}(l(t),t)^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}2\tilde{S}(t)^T P_0 \tilde{f}(t) &\leq (\gamma_1^2 + 2\gamma_2^2)\left|P_0\tilde{S}(t)\right|^2 + \frac{1}{\gamma_1^2 + 2\gamma_2^2}|\tilde{f}(t)|^2 \\ &\leq (\gamma_1^2 + 2\gamma_2^2)\left|P_0\tilde{S}(t)\right|^2 + \left|\tilde{S}(t)\right|^2 \\ &\quad + \frac{\hat{\gamma}_1^2}{\gamma_1^2 + 2\gamma_2^2}\|\tilde{\alpha}(\cdot,t)\|^2 + \frac{\gamma_2^2}{\gamma_1^2 + 2\gamma_2^2}\tilde{\alpha}(l(t),t)^2 \\ &= (\gamma_1^2 + 2\gamma_2^2)\tilde{S}(t)^T P_0^T P_0 \tilde{S}(t) + \tilde{S}(t)^T\tilde{S}(t) \\ &\quad + \frac{\hat{\gamma}_1^2}{\gamma_1^2 + 2\gamma_2^2}\|\tilde{\alpha}(\cdot,t)\|^2 + \frac{\gamma_2^2}{\gamma_1^2 + 2\gamma_2^2}\tilde{\alpha}(l(t),t)^2 \end{aligned} \tag{F.4}$$

where Young's inequality is used. Substituting (F.4) into (F.1), yields

$$\begin{aligned}\dot{V}_s(t) &\leq \tilde{S}(t)^T((\bar{A} - \mathcal{B}C_2)^T P_0 + P_0(\bar{A} - \mathcal{B}C_2))\tilde{S}(t) \\ &\quad + (\gamma_1^2 + 2\gamma_2^2)\tilde{S}^T(t)P_0^T P_0\tilde{S}(t) + \tilde{S}(t)^T\tilde{S}(t) \\ &\quad + \frac{\hat{\gamma}_1^2}{\gamma_1^2 + 2\gamma_2^2}\|\tilde{\alpha}(\cdot,t)\|^2 + \frac{\gamma_2^2}{\gamma_1^2 + 2\gamma_2^2}\tilde{\alpha}(l(t),t)^2 \\ &\leq \tilde{S}(t)^T\Big((\bar{A} - \mathcal{B}C_2)^T P_0 + P_0(\bar{A} - \mathcal{B}C_2) \\ &\quad + (\gamma_1^2 + 2\gamma_2^2)P_0^T P_0 + I^T\Big)\tilde{S}(t) \\ &\quad + \frac{\hat{\gamma}_1^2}{\gamma_1^2 + 2\gamma_2^2}\|\tilde{\alpha}(\cdot,t)\|^2 + \frac{\gamma_2^2}{\gamma_1^2 + 2\gamma_2^2}\tilde{\alpha}(l(t),t)^2. \end{aligned} \tag{F.5}$$

Recalling (119), we arrive (120).

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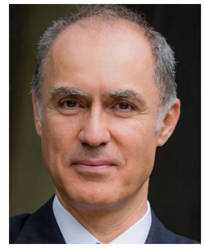

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