

# Delay-Adaptive Boundary Control of Coupled Hyperbolic PDE-ODE Cascade Systems

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**Abstract**—This article presents a delay-adaptive boundary control scheme for a  $2 \times 2$  coupled linear hyperbolic partial differential equation (PDE)-ordinary differential equation (ODE) cascade system with an unknown and arbitrarily long input delay. To construct a nominal delay-compensated control law, assuming a known input delay, a three-step backstepping design is used. To build the delay-adaptive boundary control law, the nominal control action is fed with the estimate of the unknown delay, which is generated from a batch least-squares identifier that is updated by an event-triggering mechanism that evaluates the growth of the norm of the system states. As a result of the closed-loop system, the actuator and plant states can be regulated exponentially while avoiding Zeno occurrences. The prescribed-time identification of the unknown delay is also achieved. As far as we know, this is the first delay-adaptive control result for systems governed by heterodirectional hyperbolic PDEs. The effectiveness of the proposed design is demonstrated in the control application of a deep-sea construction vessel with cable-payload oscillations and subject to input delay.

**Index Terms**—Delay-adaptive control, event-triggered control, hyperbolic partial differential equations (PDEs), least-squares identifier.

## I. INTRODUCTION

### A. Boundary Control of Coupled Hyperbolic Partial Differential Equations (PDEs)

SYSTEMS of transport PDEs appear in many physical models, including road traffic [26], [64], [65], water management systems [20], [21], [45], [46], flow of fluids in oil drilling systems [14], [27], [28], and cable vibration dynamics [53], [60]. As a result of the backstepping design [15], [52], the sliding mode control approach [40] and the proportional–integral controller design [51], the theoretical results on boundary control

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of coupled first-order linear hyperbolic PDEs have emerged in the last decade. The backstepping design was further extended to that of a  $n + 1$  system in [44], and then to a more general coupled transport PDE system where the number of PDEs in either direction is arbitrary [30]. Along the same lines, studies on the design of an adaptive estimation framework have been proposed in [2] and [3], and extended to adaptive control in [6]. However, the problem of delay-adaptive control for hyperbolic PDEs has gone unanswered in all these last developments as traditional designs based on swapping identifiers, passive identifiers, and Lyapunov functions remain difficult to exploit for such systems.

### B. Delay-Compensated Control of Finite- and Infinite-Dimensional Plants

Time delays, which are well known to be detrimental to stability [25], often exist in practical control systems. In order to compensate for arbitrarily long delays, “avant-garde” backstepping-based delay compensation techniques were first developed in [35] and [37]. Bottom-line, the input delay is converted into a transport PDE as an infinite-dimensional representation of the actuator state. For ODE plants, a PDE/ODE cascade system ensues from this substitute representation of the actuator state. The method has also been used to compensate for the effect of sensor delays that oftentimes occur in ODE plants. In comparison to many results [1], [13], [24], which only estimate plant states, the approach proposed in [35] and [37] enables estimation of both the plant and the sensor states when designing a feedback loop. A number of results considering delays that are described by complex transport actuation paths for nonlinear ODE plants were developed in [18] and [19], and the references therein.

While compensation for arbitrarily long delays is commonly available for finite-dimensional systems, only very few examples for infinite-dimensional systems were presented, where one pioneering result is [36] that is conceived using backstepping. In recent years, researchers from the PDE control community have shifted their attention to this topic, leading to many interesting developments that can be found in [34], [41], [42], [48], and [50]. By treating the delay as a transport PDE, [38] presented the design of a boundary controller for a pure wave PDE with compensation of an arbitrarily long input delay while ensuring exponential stability for the closed-loop system. For coupled heterodirectional hyperbolic PDEs, in [54], a delay-compensated control scheme was designed for a sandwich hyperbolic PDE in the presence of a sensor delay of arbitrary length. In the same spirit, Qi et al. [49] proposed a distributed input delay compensation for traffic systems governed by coupled hyperbolic PDEs (see [47] as well). In addition to the continuous-in-time control law, on the basis of the event-triggered boundary

control design of PDEs [22], [23], [55], an event-triggered delay-compensated boundary control law for coupled hyperbolic PDEs was presented in [57]. Although the above substantial results emblematic a major step in the field, the prior knowledge of the delay length is a mandatory weakening factor that mitigates their viability for many practical applications.

Lyapunov designs have been employed to develop delay-adaptive controllers for linear and nonlinear ODE plants [9], [11], [12], [39], [66], [67] via backstepping-based certainty-equivalence compensators. The primary idea behind these contributions is to estimate the unknown delay using input–output signals, and then adjust a predesigned nominal controller based on estimated parameters in order to achieve convergence. In general, compared with the other traditional parameter identifier methods, such as swapping or passive identifiers, the Lyapunov technique provides better transient performance properties. Recently, the approach has been extended to linear reaction–diffusion PDEs with a boundary or a distributed delayed input [61], [62], where asymptotic convergence results are achieved. In the realm of advancing the design approach of [61], [62], a recent result has achieved the design of Lyapunov-based delay-adaptive boundary control for a scalar integral PDE [63]. As far as we are aware, the three preceding contributions are the sole results on delay-adaptive control for PDE plants. The method in the present contribution is different with both the above delay-adaptive control results and traditional adaptive control designs for hyperbolic PDEs [6]. More precisely, our design relies on a triggered batch least-square identifier (BaLSI), a novel approach that was initially introduced in [31] and [32], which has at least two significant advantages over the traditional adaptive control approaches: guaranteeing exponential regulation of the states to zero, as well as finite-time convergence of the estimates to the true values. This method has been applied in adaptive control of a parabolic PDE [33], and first-order hyperbolic PDEs in [56], [58], and [59] with unknown plant parameters.

### C. Contributions

- 1) Different with the delay-robust stabilizing feedback control design for coupled first-order hyperbolic PDEs that achieve robustness to small delays in actuation [8], the present contribution ensures exact compensation of the arbitrarily large unknown input delay.
- 2) Exact identification of the unknown delay before the prescribed time is achieved. As a result, the exponential regulation, instead of the asymptotic one in [61] and [62], is guaranteed in the closed-loop system. Basically, after the prescribed-time identification of the unknown parameter, the delay-adaptive control signal is identical to the nominal control action (with known input delay), which ultimately improves substantially the resulting transient performance of the whole closed-loop system's dynamics.
- 3) To the best of the authors' knowledge, our result is the first delay-adaptive controller for coupled hyperbolic PDEs involving an unknown and arbitrarily large input delay. In the context of adaptive control of first-order hyperbolic PDEs with unknown transport speeds, as compared with the work in [4], [5], and [56], in our work the system

cascaded to the first-order hyperbolic PDE capturing actuation delay is a class of coupled hyperbolic PDEs-ODE systems, which is much more complicated than [4], [5], [56] where the cascaded system is a specific scalar ODE or none, and moreover, the finite-time identification is improved to the prescribed-time identification of the unknown transport speed.

### D. Organization

The rest of this article is organized as follows. The problem formulation is shown in Section II. The nominal control design is presented in Section III. The design of delay-adaptive control with piecewise-constant parameter identification is proposed in Section IV. The main result including the absence of a Zeno phenomenon, parameter convergence, and exponential regulation of the states is proved in Section V. The effectiveness of the proposed design is illustrated with a numerical simulation of a deep-sea construction vessel (DCV) in Section VI. Finally, Section VII concludes this article.

### E. Notations

We adopt the following notations.

- 1) The symbol  $\mathbb{Z}^+$  denotes the set of natural numbers including zero, and the notation  $\mathbb{N}$  for the set  $\{1, 2, \dots\}$ , i.e., the natural numbers without 0. We also use  $\mathbb{R}_+ := [0, +\infty)$ .
- 2) Let  $U \subseteq \mathbb{R}^m$  be a set with nonempty interior and let  $\Omega \subseteq \mathbb{R}$  be a set. By  $C^0(U; \Omega)$ , we denote the class of continuous mappings on  $U$ , which takes values in  $\Omega$ . By  $C^k(U; \Omega)$ , where  $k \geq 1$ , we denote the class of continuous functions on  $U$ , which have continuous derivatives of order  $k$  on  $U$  and take values in  $\Omega$ .
- 3) We use the notation  $L^2(0, 1)$  for the standard space of the equivalence class of square-integrable, measurable functions defined on  $(0, 1)$  and  $\|f\| = (\int_0^1 f(x)^2 dx)^{\frac{1}{2}} < +\infty$  for  $f \in L^2(0, 1)$ .
- 4) For an  $I \subseteq \mathbb{R}_+$ , the space  $C^0(I; L^2(0, 1))$  is the space of continuous mappings  $I \ni t \mapsto u[t] \in L^2(0, 1)$ .
- 5) Let  $u : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$  be given. We use the notation  $u[t]$  to denote the profile of  $u$  at certain  $t \geq 0$ , i.e.,  $(u[t])(x) = u(x, t)$ , for all  $x \in [0, 1]$ .

## II. PROBLEM FORMULATION

Consider the potentially open-loop unstable plant governed by the following  $2 \times 2$  linear hyperbolic PDE coupled with a linear ODE

$$\dot{X}(t) = AX(t) + Bw(0, t) \quad (1)$$

$$z_t(x, t) = -q_1 z_x(x, t) + d_1 z(x, t) + d_2 w(x, t) \quad (2)$$

$$w_t(x, t) = q_2 w_x(x, t) + d_3 z(x, t) + d_4 w(x, t) \quad (3)$$

with the boundary conditions

$$z(0, t) = CX(t) - pw(0, t) \quad (4)$$

$$w(1, t) = c_0 U(t - D) + qz(1, t) \quad (5)$$

where,  $q, q_1, q_2, d_1, d_2, d_3, d_4, c_0$ , and  $p$  are arbitrary parameters with  $q_1, q_2 > 0$  being transport speeds, and  $p \neq 0, c_0 \neq 0$ . Here, the matrix  $A, B$ , and  $C$  are known,  $z(x, t)$  and  $w(x, t)$  are the PDE state variables,  $X(t) \in \mathbb{R}^m$  is the linear ODE state,  $U$  is

the control variable, and  $D > 0$  is the indiscriminately large and unknown input delay. We assume that the initial conditions satisfy

$$z^0(x), w^0(x) \in L^2(0, 1), X^0 \in \mathbb{R}^m \quad (6)$$

and consider the following assumptions.

*Assumption 1:* The pair  $A, B$  is controllable.

*Assumption 2:* Parameters  $p, q$  satisfy

$$|pq|e^{\max\left\{\frac{2d_4}{q_2}, \frac{2d_1}{q_1}\right\}} < \frac{1}{\sqrt{2}}. \quad (7)$$

*Assumption 3:* The bounds of the unknown input delay  $D$  are known and arbitrary, i.e.,

$$0 < \underline{D} \leq D \leq \bar{D} \quad (8)$$

where positive constants  $\underline{D}$  and  $\bar{D}$  are arbitrary.

Our goal is to design a delay-adaptive boundary control action,  $U(t)$ , that exponentially regulates the system (1)–(5) despite the presence of an unknown delay  $D$  whose length is arbitrary. The plant (1)–(5) can be used to model cable-payload oscillations in DCV, which are to be suppressed for the purpose of accurate placement of the equipment to be installed on the sea floor. From this application perspective, large-distance signal transmission in the water through a set of acoustics devices and the actuation of the hydraulic actuator for the ship-mounted crane are subject to delays, which are considered as an unknown delay in the control input, the cable vibration dynamics are governed by the  $2 \times 2$  hyperbolic PDE, and the vibration dynamics of the cage are captured by the ODE system.

### III. NOMINAL DELAY-COMPENSATED CONTROL DESIGN

In order to design the nominal control law, we first construct an infinite-dimensional representation of the actuator state by converting the delayed input into transport PDE actuation dynamics. Define a new variable  $v(x, t)$  as

$$v(x, t) = \begin{cases} U(t - Dx) & \text{if } t - Dx \geq 0 \\ 0 & \text{if } t - Dx < 0. \end{cases}$$

Then, (5) is rewritten as

$$w(1, t) = c_0 v(1, t) + qz(1, t) \quad (9)$$

$$v_t(x, t) = -\frac{1}{D}v_x(x, t) \quad \forall x \in (0, 1) \quad (10)$$

$$v(0, t) = U(t) \quad (11)$$

$$v(x, 0) = 0 \quad \forall x \in (0, 1], \quad (12)$$

for  $t \in [0, \infty)$ . Now, resulting from the new representation of the actuator state, the function  $U(t)$ , which is defined as the boundary condition (11) of the transport equation (10), is the delay-free control input to be designed for the hyperbolic PDE-PDE-ODE cascade system consisting of (1)–(4) combined with (9)–(12).

#### A. First Step: Backstepping Transformation for the $2 \times 2$ Coupled Hyperbolic PDE-ODE

We introduce the following backstepping transformation [43] in order to remove the in-domain coupling destabilizing terms from the  $2 \times 2$  hyperbolic PDE system consisting of (2) and (3), and make the ODE system matrix Hurwitz

$$\alpha(x, t) = z(x, t) - \int_0^x \phi(x, y)z(y, t)dy$$

$$- \int_0^x \varphi(x, y)w(y, t)dy - \gamma(x)X(t) \quad (13)$$

$$\beta(x, t) = w(x, t) - \int_0^x \Psi(x, y)z(y, t)dy \\ - \int_0^x \Phi(x, y)w(y, t)dy - \lambda(x)X(t) \quad (14)$$

whose inverse is

$$z(x, t) = \alpha(x, t) - \int_0^x \bar{\phi}(x, y)\alpha(y, t)dy \\ - \int_0^x \bar{\varphi}(x, y)\beta(y, t)dy - \bar{\gamma}(x)X(t) \quad (15)$$

$$w(x, t) = \beta(x, t) - \int_0^x \bar{\Psi}(x, y)\alpha(y, t)dy \\ - \int_0^x \bar{\Phi}(x, y)\beta(y, t)dy - \bar{\lambda}(x)X(t) \quad (16)$$

to convert (1)–(4) and (9) into

$$\dot{X}(t) = A_m X(t) + B\beta(0, t) \quad (17)$$

$$\alpha(0, t) = -p\beta(0, t) \quad (18)$$

$$\alpha_t(x, t) = -q_1\alpha_x(x, t) + d_1\alpha(x, t) \quad (19)$$

$$\beta_t(x, t) = q_2\beta_x(x, t) + d_4\beta(x, t) \quad (20)$$

$$\beta(1, t) = c_0 v(1, t) + q\alpha(1, t) + (\bar{\lambda}(1) - q\bar{\gamma}(1))X(t) \\ + \int_0^1 (\bar{\Psi}(1, y) - q\bar{\phi}(1, y))\alpha(y, t)dy \\ + \int_0^1 (\bar{\Phi}(1, y) - q\bar{\varphi}(1, y))\beta(y, t)dy. \quad (21)$$

The gain vector  $K$  is selected so that

$$A_m = A + BK^T \quad (22)$$

is Hurwitz.

The conditions on the kernels  $\phi(x, y)$ ,  $\varphi(x, y)$ ,  $\gamma(x)$ ,  $\Psi(x, y)$ ,  $\Phi(x, y)$ , and  $\lambda(x)$ , and  $\bar{\phi}(x, y)$ ,  $\bar{\varphi}(x, y)$ ,  $\bar{\gamma}(x)$ ,  $\bar{\Psi}(x, y)$ ,  $\bar{\Phi}(x, y)$ , and  $\bar{\lambda}(x)$  in the backstepping transformations (13)–(16), which are obtained by matching the original system (1)–(5) and the intermediate system (17)–(21), are shown in the part 1 of Appendix A, and the well-posedness of the kernel conditions has been proved in [43, Th. 4.1].

#### B. Second Step: Transformation of the Actuator States

With the purpose of removing the integral terms and ODE state  $X(t)$  from the PDE boundary condition (21), we define the following change of coordinate:

$$u(x, t) = v(x, t) + \int_0^1 K_1(x, y)\alpha(y, t)dy \\ + \int_0^1 K_2(x, y)\beta(y, t)dy + \eta(x)X(t) \quad (23)$$

which enables one to map the actuator dynamics given by (10), (11), and (21) into the following equations:

$$\beta(1, t) = c_0 u(1, t) + q\alpha(1, t) \quad (24)$$

$$u_t(x, t) = -du_x(x, t) + q_2 K_2(x, 1)c_0 u(1, t) \quad (25)$$

$$u(0, t) = U(t) + \int_0^1 K_1(0, y)\alpha(y, t)dy$$

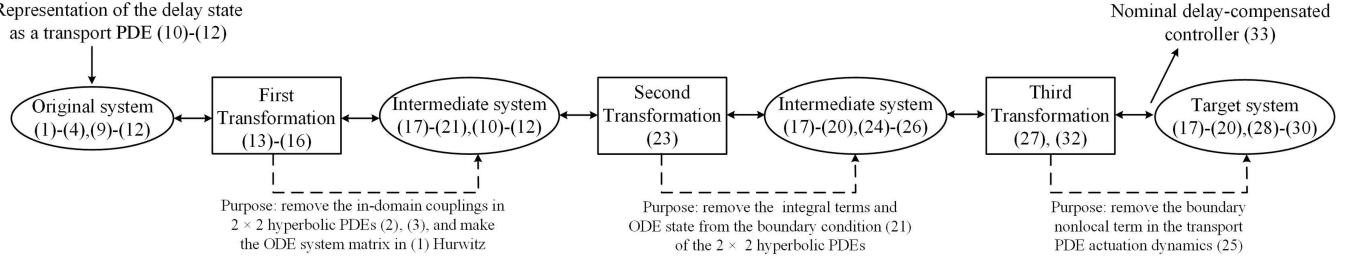


Fig. 1. Flow diagram of the nominal delay-compensated control design.

$$+ \int_0^1 K_2(0, y) \beta(y, t) dy + \eta(0) X(t) \quad (26)$$

where

$$d = \frac{1}{D}.$$

The detailed computation and conditions of the kernels  $K_1(x, y), K_2(x, y), \eta(x)$  are given in the part 2 of Appendix A.

### C. Third Step: Backstepping Transformation for the Resulting $u$ -PDE

To remove the boundary nonlocal term  $q_2 K_2(x, 1) c_0 u(1, t)$  in the transport PDE (25), we apply the following mapping:

$$u(x, t) = \hat{u}(x, t) + \int_x^1 R(x, y) \hat{u}(y, t) dy \quad (27)$$

which converts (24)–(26) into

$$\beta(1, t) = c_0 \hat{u}(1, t) + q \alpha(1, t) \quad (28)$$

$$\hat{u}_t(x, t) = -d \hat{u}_x(x, t) \quad (29)$$

$$\hat{u}(0, t) = 0 \quad (30)$$

with the nominal control input defined as

$$\begin{aligned} U(t) = & - \int_0^1 K_1(0, y; D) \alpha(y, t) dy - \int_0^1 K_2(0, y; D) \beta(y, t) dy \\ & + \int_0^1 R(0, y; D) \hat{u}(y, t) dy - \eta(0; D) X(t). \end{aligned} \quad (31)$$

The conditions of the kernel  $R(x, y)$  are shown in the part 3 of Appendix A. Writing  $D$  after “;” in (31) emphasizes the fact that these functions are parameterized by the delay  $D$ . The inverse transformation of (27) can be found as

$$\hat{u}(x, t) = u(x, t) + \int_x^1 P(x, y) u(y, t) dy \quad (32)$$

where the conditions of  $P(x, y)$  are given in the part 3 of Appendix A as well.

### D. Stability Result of Nominal Delay-Compensated Control

The flow diagram of the nominal delay-compensated control is shown in Fig. 1. In a nutshell, the prior transformations convert the original system that consists of (1)–(4) and (9)–(12) into the target system that consists of (17)–(20) and (28)–(30). The nominal control input (31) is rewritten with respect to the original state variables as follows:

$$U(t) = \int_0^1 M_1(y; D) z(y, t) dy + \int_0^1 M_2(y; D) w(y, t) dy$$

$$+ \int_0^1 M_3(y; D) v(y, t) dy + M_4(D) X(t) \quad (33)$$

where the controller gains  $M_i, i = 1, \dots, 4$  are given in Appendix B, which includes the delay  $D$ .

The stability result of the nominal delay-compensated control is stated as follows.

**Theorem 1:** For the known delay  $D$ , with arbitrary initial data  $(z[0], w[0])^T \in L^2(0, 1)$ ,  $X(0) \in \mathbb{R}^m$ , considering the closed-loop system consisting of the plant (1)–(4), (9)–(12), and the nominal controller (33), the exponential stability of the closed-loop system is obtained in the sense that there exist positive constants  $\Upsilon, \lambda_1$  such that

$$\Omega(t) \leq \Upsilon \Omega(0) e^{-\lambda_1 t}, \quad t \geq 0 \quad (34)$$

where  $\Omega(t)$  is defined as

$$\Omega(t) = \|z[t]\|^2 + \|w[t]\|^2 + \|v[t]\|^2 + |X(t)|^2. \quad (35)$$

**Proof:** Define Lyapunov function  $V(t)$  as

$$\begin{aligned} V(t) = & \frac{r_d}{2} X^T(t) P_1 X(t) + \frac{r_a}{2} \int_0^1 e^{\delta x} \beta(x, t)^2 dx \\ & + \frac{1}{2} \int_0^1 e^{-\delta x} \alpha(x, t)^2 dx + \frac{r_c}{2} \int_0^1 e^{-x} \hat{u}(x, t)^2 dx \end{aligned} \quad (36)$$

where a positive definite matrix  $P_1 = P_1^T$  is the solution to the Lyapunov equation  $A_m^T P_1 + P_1 A_m = -Q_1$  for some  $Q_1 = Q_1^T > 0$ , and where  $\delta, r_a, r_c, r_d$  satisfy

$$e^{-2\max\{\frac{2d_4}{q_2}, \frac{2d_1}{q_1}\}} > e^{-2\delta} > 2p^2 q^2 \quad (37)$$

$$\frac{q_1}{2q_2 q^2} e^{-2\delta} \geq r_a > \frac{p^2 q_1}{q_2} \quad (38)$$

$$r_c \geq 2\bar{D} q_2 r_a e^\delta c_0^2 \quad (39)$$

$$0 < r_d \leq \frac{\lambda_{\min}(Q_1)}{2|P_1 B|^2} (q_2 r_a - p^2 q_1). \quad (40)$$

Please note that  $e^{-2\max\{\frac{2d_4}{q_2}, \frac{2d_1}{q_1}\}} > 2p^2 q^2$  holds in (37) under Assumption 2, and  $\frac{q_1}{2q_2 q^2} e^{-2\delta} > \frac{p^2 q_1}{q_2}$  holds in (38) due to the right inequality in (37), which means the existence of  $\delta, r_a$  satisfying (37) and (38). It is then straightforward to obtain  $r_c, r_d$  by (39) and (40), where the positiveness of the right-hand side of (40) is ensured by the right inequality in (38). Therefore, there exists a solution  $\delta, r_a, r_c$ , and  $r_d$  satisfying (37)–(40).

According to Appendix D where the relationship (161) between the norms of states in the original and target systems is obtained, we have

$$\xi_1 \xi_3 \Omega(t) \leq V(t) \leq \xi_2 \xi_4 \Omega(t) \quad (41)$$

where  $\xi_1$  and  $\xi_2$  are given in (162) and (163) in Appendix D, and

$$\xi_3 = \frac{1}{2} \min \{r_d \lambda_{\min}(P_1), r_a, e^{-\delta}, r_c e^{-1}\} \quad (42)$$

$$\xi_4 = \frac{1}{2} \max \{r_d \lambda_{\max}(P_1), r_a e^{\delta}, 1, r_c\} \quad (43)$$

where  $\lambda_{\min}(P_1)$  is the smallest eigenvalue of  $P_1$ .

Taking the derivative of (36) along (17)–(20) and (28)–(30), and applying the Young's inequality and the Cauchy–Schwarz inequality, the following estimate holds for all  $t \geq 0$ :

$$\begin{aligned} \dot{V}(t) \leq & -\frac{r_d}{4} \lambda_{\min}(Q_1) |X(t)|^2 \\ & - \left( \frac{1}{2} q_1 e^{-\delta} - q_2 r_a e^{\delta} q^2 \right) \alpha(1, t)^2 \\ & - \left( \frac{1}{2} q_2 r_a - \frac{r_d |P_1 B|^2}{\lambda_{\min}(Q_1)} - \frac{p^2 q_1}{2} \right) \beta(0, t)^2 \\ & - r_a \left( \frac{1}{2} \delta q_2 - d_4 \right) \int_0^1 e^{\delta x} \beta(x, t)^2 dx \\ & - \left( \frac{1}{2} \delta q_1 - d_1 \right) \int_0^1 e^{-\delta x} \alpha(x, t)^2 dx \\ & - \left( \frac{r_c}{2D} - q_2 r_a e^{\delta} c_0^2 \right) \hat{u}(1, t)^2 \\ & - \frac{r_c}{2D} \int_0^1 e^x \hat{u}(x, t)^2 dx. \end{aligned} \quad (44)$$

Recalling conditions (37)–(39) on  $\delta, r_a, r_c$ , and  $r_d$ , there exists a sufficiently small positive constant  $\lambda_1$ , such that

$$\dot{V}(t) \leq -\lambda_1 V(t) \quad (45)$$

where

$$\lambda_1 = \min \left\{ \frac{\lambda_{\min}(Q_1)}{2\lambda_{\max}(P_1)}, \delta q_2 - 2d_4, \delta q_1 - 2d_1, \frac{1}{D} \right\} > 0. \quad (46)$$

Recalling (41), we then obtain (34) where the positive constant  $\Upsilon$  is given as

$$\Upsilon = \frac{\xi_2 \xi_4}{\xi_1 \xi_3}. \quad (47)$$

The proof of the theorem is complete.  $\blacksquare$

Next, we will design a delay-adaptive controller considering the nominal control action (33) fed with an estimate  $\hat{D}$  that is given by an update law resulting from a triggered BaLSI of the unknown delay  $D$ .

#### IV. DELAY-ADAPTIVE CONTROL DESIGN

Before presenting the controller, we propose the design of a triggered batch least-squares identifier for the unknown delay, in the following two sections.

##### A. Triggering Mechanism

The triggering mechanism for the batch least-squares identifier is defined as

$$t_{i+1} = \begin{cases} \min \left\{ \inf \left\{ t > t_i : \Omega(t) = (1+a) \hat{\Upsilon}(\hat{D}(t_i)) \Omega(t_i) \right\}, \right. \\ \left. t_i + T \right\}, & \text{for } \Omega(t_i) \neq 0 \\ t_i + T, & \text{for } \Omega(t_i) = 0 \end{cases} \quad (48)$$

where the positive design parameter  $a$  is free, another positive design parameter

$$T \leq \frac{T_f}{2} \quad (49)$$

is the maximum dwell time between two adjacent triggering time instants, and the free design parameter  $T_f > 0$  is the prescribed time of identifying the unknown delay. The function  $\Omega(t)$  is given in (35). The function  $\hat{\Upsilon}(\hat{D}(t_i)) \geq 1$  is the overshoot coefficient that is associated with the system transient and is obtained by replacing the unknown  $D$  with  $\hat{D}(t_i)$  in  $\Upsilon$  which is defined in (47) [note that  $\xi_1$  and  $\xi_2$  in (47) depend on the delay  $D$  through the delay-dependent kernel functions  $K_1, K_2, \eta, R, P$  included in (162) and (163) in Appendix D. See Appendix D for further details].

##### B. Least-Squares Identifier for the Unknown Delay

Now, we design the identifier which stands as the update law of the estimated delay  $\hat{D}$ . According to (10), for  $\tau > 0$  and  $n = 1, 2, \dots$ , the following equality holds:

$$D \frac{d}{d\tau} \int_0^1 \sin(x\pi n) v(x, \tau) dx = \pi n \int_0^1 \cos(x\pi n) v(x, \tau) dx. \quad (50)$$

Integrating (50) from 0 to  $t$ , yields

$$D \int_0^1 \sin(x\pi n) v(x, t) dx = \pi n \int_0^t \int_0^1 \cos(x\pi n) v(x, \tau) dx d\tau \quad (51)$$

where (12) has been recalled. Straightforwardly, (51) can be written as

$$f_n(t) = D g_n(t) \quad (52)$$

where

$$f_n(t) = \pi n \int_0^t \int_0^1 \cos(x\pi n) v(x, \tau) dx d\tau \quad (53)$$

$$g_n(t) = \int_0^1 \sin(x\pi n) v(x, t) dx \quad (54)$$

for  $n \in \mathbb{N}$ . Define the function  $h_{i,n}$  by the formula

$$h_{i,n}(\ell) = \int_{\mu_{i+1}}^{t_{i+1}} (f_n(t) - \ell g_n(t))^2 dt, \quad i \in \mathbb{Z}^+, n \in \mathbb{N} \quad (55)$$

and time instant  $\mu_{i+1}$  as

$$\mu_{i+1} := \min \{t_g : g \in \{0, \dots, i\}, t_g \geq t_{i+1} - \tilde{N}T\} \quad (56)$$

where the positive integer  $\tilde{N} \geq 1$  is a free design parameter (in practice, a larger  $\tilde{N}$  means a bigger set of data used in the least-squares identifier, which makes the identifier more robust with respect to measurement errors), and where the positive constant  $T$  is the maximum dwell time according to (48). From (52), one can deduce that the function  $h_{i,n}(\ell)$  in (55) has a global minimum  $h_{i,n}(D) = 0$ . Then, using Fermat's theorem (vanishing gradient at extrema), the following matrix equation hold for every  $i \in \mathbb{Z}^+$  and  $n \in \mathbb{N}$ :

$$H_n(\mu_{i+1}, t_{i+1}) = G_n(\mu_{i+1}, t_{i+1}) D \quad (57)$$

where

$$H_n(\mu_{i+1}, t_{i+1}) = \int_{\mu_{i+1}}^{t_{i+1}} g_n(t) f_n(t) dt \quad (58)$$

$$G_n(\mu_{i+1}, t_{i+1}) = \int_{\mu_{i+1}}^{t_{i+1}} g_n(t)^2 dt. \quad (59)$$

Indeed, (57) is obtained by differentiating the functions  $h_{i,n}(\ell)$  defined by (55) with respect to  $\ell$ , and evaluating the derivative (zero) at the global minimum  $\ell = D$ . Using (57)–(59), the following delay identifier is constructed:

$$\begin{aligned}\hat{D}(t_{i+1}) &= \operatorname{argmin}\{|\ell - \hat{D}(t_i)|^2 : \underline{D} \leq \ell \leq \bar{D}\}, \\ H_n(\mu_{i+1}, t_{i+1}) &= G_n(\mu_{i+1}, t_{i+1})\ell, \quad n = 1, 2, \dots, \quad i \in \mathbb{Z}^+.\end{aligned}\quad (60)$$

*Remark 1 (Implementation of the identifier):* Implementation of the identifier begins with calculating  $H_n(\mu_{i+1}, t_{i+1})$ ,  $G_n(\mu_{i+1}, t_{i+1})$  from  $n = 1$ ,  $i = 0$ , i.e.,  $H_1(\mu_1, t_1)$ ,  $G_1(\mu_1, t_1)$ , using (53), (54), (58), and (59). If  $G_1(\mu_1, t_1) \neq 0$ , it implies that  $\ell$  belongs to a singleton set, i.e.,  $\ell = \frac{H_1(\mu_1, t_1)}{G_1(\mu_1, t_1)}$ . It is followed that the output of the identifier (60) at  $t_1$  is  $\hat{D}(t_1) = \frac{H_1(\mu_1, t_1)}{G_1(\mu_1, t_1)}$ . If  $G_1(\mu_1, t_1) = 0$ , we continue to calculate  $H, G$  with  $n = 2$ ,  $i = 0$ , i.e.,  $H_2(\mu_1, t_1)$ ,  $G_2(\mu_1, t_1)$ , and then evaluate the value of  $G_2(\mu_1, t_1)$ . Similarly, if  $G_2(\mu_1, t_1) \neq 0$ , the output of the identifier at  $t_1$  is  $\hat{D}(t_1) = \frac{H_2(\mu_1, t_1)}{G_2(\mu_1, t_1)}$ . If  $G_2(\mu_1, t_1) = 0$ , then move to calculate the case of  $n = 3$ ,  $i = 0$ , i.e.,  $H_3(\mu_1, t_1)$ ,  $G_3(\mu_1, t_1)$ . Repeating the above steps, until we find a  $G_n(\mu_1, t_1) \neq 0$  for a certain  $n$ , the output of the identifier at  $t_1$  is  $\hat{D}(t_1) = H_n(\mu_1, t_1)/G_n(\mu_1, t_1)$ . For saving the computation time, we can set an upper limit  $\bar{n}$  for  $n$ . That is, if  $G_n(\mu_1, t_1) = 0$  for all  $n = 1, \dots, \bar{n}$ , we then stop the seeking at the updating time  $t_1$  and consider  $\ell$  belongs to the original set  $\{\ell \in \mathbb{R} : \underline{D} \leq \ell \leq \bar{D}\}$ , which leads to the output of the identifier is equal to the estimate at the last time instant, i.e.,  $\hat{D}(t_1) = \hat{D}(t_0)$ , according to (60). The same computation process is followed for the subsequent updating time instants  $t_2, t_3, \dots$ . For many practical applications, such as simulating a DCV, locating nonzero values of  $G_n(\mu_{i+1}, t_{i+1})$  is a straightforward task following the algorithm described above.

Note that even though the actuator states  $v(x, t)$  are measurable in this full-state feedback case, for the delay estimation, one cannot adopt the “naive” method—that is, taking the time and spatial derivatives of the signal  $v(x, t)$  to calculate  $d$  in (10) straightforwardly—because of the following two reasons: 1) taking the time derivative of the measured signals always leads to the undesired noise amplification in practice; 2) the possible zero values of  $v_t(x, t)$  accompanied with the unknown delay  $D$  will engender singularity.

### C. Delay-Adaptive Controller

With the sequence of time instants  $\{t_i \geq 0\}_{i=0}^{\infty}$ ,  $i \in \mathbb{Z}^+$  determined by the triggering mechanism (48), and the parameter identifier (60), the delay-compensated adaptive control algorithm  $U_d(t)$  on  $t \in [t_i, t_{i+1}]$  is designed as

$$U_d(t) = \begin{cases} r \left( \sin(\omega(t - t_i + \frac{\pi}{2\omega})) - 1 \right) & \text{if } t_i > T_f - 2T \text{ and } t_{i-1} \leq T_f - 2T \text{ and } i \geq 1 \\ & \text{and } U_d(t) \equiv 0 \text{ on } t \in [0, t_i] \\ \int_0^1 M_1(y; \hat{D}(t_i))z(y, t)dy \\ + \int_0^1 M_2(y; \hat{D}(t_i))w(y, t)dy \\ + \int_0^1 M_3(y; \hat{D}(t_i))v(y, t)dy \\ + M_4(\hat{D}(t_i))X(t), & \text{otherwise} \end{cases} \quad (61a)$$

where  $T_f - 2T \geq 0$  according to (49). The (61b) is the result of replacing the unknown delay  $D$  in the nominal continuous-in-

time feedback (33) with the estimate  $\hat{D}(t_i)$  that is generated with the triggered batch least-squares identifier (60). Defining the time instant  $t_i$  satisfying the condition in (61a) as  $t_z$ , (61a) is an excitation implemented in a time interval  $[t_z, t_{z+1})$  once  $U_d$  identically zero on  $[0, t_z)$  is detected, to avoid the case that  $U_d(t)$  is identically zero on  $t \in [0, t_{z+1})$ , whose purpose is to ensure the exact identification of the unknown delay in the prescribed time  $T_f$ , which will be clear in the proof of Lemma 5. The nonzero constant  $r, \omega$  in (61a) are free design parameters. Some guidelines about choosing the free design parameters  $r, \omega$ , and  $T_f$  from the practical point of view is given in Remark 2.

*Remark 2 (Selections of free  $r, \omega, T_f$  in practice):* The constant  $r$  can be chosen small enough in practice to reduce the effect of the excitation (61a) on the control performance. The frequency  $\omega$  in (61a) should be selected away from the natural inherent frequency of the plant to avoid the appearance of syntony. The prescribed identification time  $T_f$ , together with the maximum dwell time  $T$ , are positively related to the amount of the measurement data used in parameter estimation. The larger  $T_f, T$  would improve the robustness to the sensor measurement error but prolong the time till exact parameter identification. On the contrary, the smaller  $T_f, T$  contributes to the fast identification of the unknown delay, however, the robustness to the measurement error may be reduced. How to improve the robustness to the sensor measurement error under a short prescribed identification time  $T_f, T$  is our future work.

*Proposition 1 (Existence of solution in an interval):* For every  $(z[t_i], w[t_i], v[t_i])^T \in L^2((0, 1); \mathbb{R}^3)$ ,  $X(t_i) \in \mathbb{R}^m$ , there exists a unique (weak) solution  $((z, w, v)^T, X) \in C^0([t_i, t_{i+1}]; L^2(0, 1); \mathbb{R}^3) \times C^0([t_i, t_{i+1}]; \mathbb{R}^m)$  to the system (1)–(4), (9)–(12), and (61).

*Proof:* The proof is shown in Appendix C.  $\blacksquare$

## V. MAIN RESULT

Before presenting the main theorem, we propose the following technical lemmas, where when we say that  $v(x, t)$  is equal to zero for  $x \in [0, 1], t \in [\mu_{i+1}, t_{i+1}]$ , or not identically zero on the same domain, we mean except possibly for finitely many discontinuities of the functions  $v(x, t)$ . These discontinuities are isolated curves in the rectangle  $[0, 1] \times [\mu_{i+1}, t_{i+1}]$ .

*Lemma 1 ( $G_n(\mu_{i+1}, t_{i+1}) = 0$ ):* The sufficient and necessary condition of  $G_n(\mu_{i+1}, t_{i+1}) = 0$  for all  $n \in \mathbb{N}$  is  $v[t] = 0$  on  $t \in [\mu_{i+1}, t_{i+1}]$ .

*Proof: Necessity:* If  $G_n(\mu_{i+1}, t_{i+1}) = 0$  for all  $n \in \mathbb{N}$ , then the definition (59) in conjunction with continuity of  $g_n(t)$  for  $t \in [\mu_{i+1}, t_{i+1}]$  [because of the definition (54) and the fact that  $v \in C^0([t_i, t_{i+1}]; L^2(0, 1))$  in Proposition 1] implies

$$g_n(t) = 0, \quad t \in [\mu_{i+1}, t_{i+1}]. \quad (62)$$

According to the definition (54), the (62) implies

$$\int_0^1 \sin(x\pi n) v(x, t) dx = 0, \quad t \in [\mu_{i+1}, t_{i+1}] \quad (63)$$

for all  $n \in \mathbb{N}$ . Since the set  $\{\sqrt{2} \sin(x\pi n) : n = 1, 2, \dots\}$  is an orthonormal basis of  $L^2(0, 1)$ , we have  $v[t] = 0$  for  $t \in [\mu_{i+1}, t_{i+1}]$ .

*Sufficiency:* If  $v[t] = 0$  on  $t \in [\mu_{i+1}, t_{i+1}]$ , then  $G_n(\mu_{i+1}, t_{i+1}) = 0$  for all  $n \in \mathbb{N}$  is obtained directly by recalling (59) and (54).  $\blacksquare$

*Lemma 2 (Identifier properties at  $t_{i+1}$ ):* For the adaptive estimates defined by (60), the following statements hold:

- 1) If  $v[t]$  is not identically zero for  $t \in [\mu_{i+1}, t_{i+1}]$ , then  $\hat{D}(t_{i+1}) = D$ .
- 2) If  $v[t]$  is identically zero for  $t \in [\mu_{i+1}, t_{i+1}]$ , then  $\hat{D}(t_{i+1}) = \hat{D}(t_i)$ .

*Proof:* Define a set

$$S_i = \left\{ \underline{D} \leq \ell \leq \bar{D} : H_n(\mu_{i+1}, t_{i+1}) = G_n(\mu_{i+1}, t_{i+1})\ell, n = 1, 2, \dots \right\}. \quad (64)$$

From (57), we know that  $D \in S_i$ . If  $S_i$  is a singleton, it is nothing else but the generated adaptive estimate  $\hat{D}(t_{i+1})$  by (60), which is equal to the true delay  $D$ .

- 1) If  $v[t]$  is not identically zero for  $t \in [\mu_{i+1}, t_{i+1}]$ , recalling Lemma 1, there exists  $n \in \mathbb{N}$  such that  $G_n(\mu_{i+1}, t_{i+1}) \neq 0$ . Now defining the index set  $I$  as the set of all  $n \in \mathbb{N}$  with  $G_n(\mu_{i+1}, t_{i+1}) \neq 0$ , then (64) implies that

$$S_i = \left\{ \ell = \frac{H_n(\mu_{i+1}, t_{i+1})}{G_n(\mu_{i+1}, t_{i+1})}, n \in I \right\}$$

is a singleton, and therefore from (60) we get  $\hat{D}(t_{i+1}) = D$ .

- 2) If  $v[t]$  is identically zero on  $t \in [\mu_{i+1}, t_{i+1}]$ , according to (53), (54), (58), (59), one obtains

$$G_n(\mu_{i+1}, t_{i+1}) = H_n(\mu_{i+1}, t_{i+1}) = 0, n \in \mathbb{N}$$

and it follows that  $S_i = \{\underline{D} \leq \ell \leq \bar{D}\}$ . Then, from (60) one arrives at  $\hat{D}(t_{i+1}) = \hat{D}(t_i)$ .

The proof is complete.  $\blacksquare$

*Lemma 3 (Identifier properties for  $t \in [t_i, \lim_{k \rightarrow \infty}(t_k)]$ ):*

If  $\hat{D}(t_i) = D$  for certain  $i \in \mathbb{Z}^+$ , then  $\hat{D}(t) = D$  for all  $t \in [t_i, \lim_{k \rightarrow \infty}(t_k)]$ .

*Proof:* According to Lemma 2, we have that  $\hat{D}(t_{i+1})$  is equal to either  $D$  or  $\hat{D}(t_i)$ . Therefore, if  $\hat{D}(t_i) = D$ , then  $\hat{D}(t_{i+1}) = D$ . Repeating this process, we then have  $\hat{D}(t) = D$  for all  $t \in [t_i, \lim_{k \rightarrow \infty}(t_k)]$ . The proof is complete.  $\blacksquare$

*Lemma 4 (Existence of a minimum dwell-time):* There exists a positive constant  $\tau_d$  such that  $t_{i+1} - t_i \geq \tau_d$  for all  $i \in \mathbb{Z}^+$ .

*Proof:* The result is established by discussing the following two cases.

- 1) *Case 1:* The exact identification has not been achieved for  $t \in [0, t_i]$ . According to Lemmas 2 and 3, we know  $\hat{D}(t) \equiv \hat{D}(0)$  on  $t \in [0, t_{i+1}]$ . Recalling Proposition 1, we obtain that  $\Omega(t)$  is continuous on  $t \in [t_i, t_{i+1}]$ , with possible finite nondifferentiable points (though it is differentiable from the left and from the right, i.e., the left and right derivatives are finite, at those points). Denoting the maximum rate of change of  $\Omega(t)$  on  $t \in (t_i, t_{i+1})$  as  $V_i$ , that is,  $V_i = \max\{\max_{t \in (t_i, t_{i+1})/I_i} |\dot{\Omega}(t)|, A_i\}$  where  $I_i$  is the set of those possible finite nondifferentiable points, and where the set  $A_i$  is the absolute values of left and right derivatives at the points in  $I_i$ . Recalling the triggering mechanism (48), the lower bound  $\underline{\tau}_i$  of the dwell time is given by

$$\begin{aligned} \underline{\tau}_i = \\ \begin{cases} \min \left\{ \frac{((1+a)\hat{\Upsilon}(\hat{D}(0))-1)\Omega(t_i)}{V_i}, T \right\} > 0, & \text{if } \Omega(t_i) \neq 0 \\ T, & \text{if } \Omega(t_i) = 0. \end{cases} \end{aligned} \quad (65)$$

- 2) *Case 2:* The exact identification has been achieved for  $[0, t_i]$ . In this case, we have  $t_{i+1} - t_i = T$ . We prove this as follows. Once the exact delay identification is achieved, the delay-adaptive control input is identical to the nominal delay-compensated control input in Section III. When  $\Omega(t_i) \neq 0$ , we have that  $\Omega(t) \leq \Upsilon\Omega(t_i)$  for  $t_i \leq t \leq t_{i+1}$  according to Theorem 1. It follows from  $\hat{\Upsilon}(\hat{D}(t_i)) = \hat{\Upsilon}(D) = \Upsilon$  that  $\Omega(t) < (1+a)\hat{\Upsilon}(\hat{D}(t_i))\Omega(t_i)$  for  $t_i \leq t \leq t_{i+1}$ . Thus  $t_{i+1} - t_i = T$  according to (48). When  $\Omega(t_i) = 0$ , we straightforwardly have  $t_{i+1} - t_i = T$  according to the second equation in (48) and therefore,  $t_{i+1} - t_i = T$ .

The lemma is thus obtained.  $\blacksquare$

*Corollary 1 (Well-posedness of the closed-loop system):* No Zeno phenomenon occurs, i.e.,  $\lim_{i \rightarrow \infty} t_i = +\infty$ , and the closed-loop system is well-posed in the sense that for every  $(z[0], w[0])^T \in L^2((0, 1); \mathbb{R}^2)$ ,  $X(0) \in \mathbb{R}^m$ , and  $\hat{D}(0) \in [\underline{D}, \bar{D}]$ , there exists a unique (weak) solution  $((z, w, v)^T, X) \in C^0(\mathbb{R}_+; L^2(0, 1); \mathbb{R}^3) \times C^0(\mathbb{R}_+; \mathbb{R}^m)$ , and  $\hat{D}(t) \in \{\ell \in \mathbb{R} : \underline{D} \leq \ell \leq \bar{D}\}$  for  $t \in [0, \infty)$ , to the system consisting of (1)–(4), (9)–(12), (60), and (61).

*Proof:* Recalling Lemma 4, we have that

$$t_i \geq \tau_d i, \quad i \in \mathbb{Z}^+$$

where  $\tau_d > 0$ , that is

$$\lim_{i \rightarrow \infty} (t_i) = +\infty \quad (66)$$

which implies a solution defined on  $\mathbb{R}_+$  in the subsequent analysis.

From the initial data  $(z[0], w[0])^T \in L^2((0, 1); \mathbb{R}^2)$ ,  $X(0) \in \mathbb{R}^m$  and (12), recalling the result in Proposition 1 for  $i = 0$ , it follows that  $((z, w, v)^T, X) \in C^0([t_0, t_1]; L^2(0, 1); \mathbb{R}^3) \times C^0([t_0, t_1]; \mathbb{R}^m)$ , which implies  $(z[t_1], w[t_1], v[t_1])^T \in L^2((0, 1); \mathbb{R}^3)$ ,  $X(t_1) \in \mathbb{R}^m$ . Recalling the result in Proposition 1 for  $i = 1$ , together with the solution obtained for  $[t_0, t_1]$ , we have that  $((z, w, v)^T, X) \in C^0([t_0, t_2]; L^2(0, 1); \mathbb{R}^3) \times C^0([t_0, t_2]; \mathbb{R}^m)$ . Repeating the above steps, we obtain that  $((z, w, v)^T, X) \in C^0([t_0, t_i]; L^2(0, 1); \mathbb{R}^3) \times C^0([t_0, t_i]; \mathbb{R}^m)$  for  $i \in \mathbb{N}$ . Applying (66), we thus have  $((z, w, v)^T, X) \in C^0([\mathbb{R}_+; L^2(0, 1); \mathbb{R}^3] \times C^0(\mathbb{R}_+; \mathbb{R}^m)$ . It is straightforwardly obtained from (60) that  $\hat{D}(t) \in [\underline{D}, \bar{D}]$  if  $\hat{D}(0) \in [\underline{D}, \bar{D}]$ .

Corollary 1 is thus obtained.  $\blacksquare$

*Lemma 5 (Finite-time convergence of the update law):* The estimate  $\hat{D}$  converges to the true value no latter than  $T_f$ , i.e.,

$$\hat{D}(t) = D \quad \forall t \in [t_f, \infty) \quad (67)$$

where  $0 < t_f \leq T_f$ .

*Proof:* According to (61), we conclude that the control input  $U_d(t)$  is not identically zero on  $t \in [0, t_{z+1}]$  where the time instant  $t_z$  satisfying the condition in (61a) is denoted as  $t_z$ . There exists a time instant  $t_f \leq t_{z+1}$  ( $f > 0$ ) such that  $U_d(t)$  is not identically zero on  $t \in [\mu_f, t_f]$ . Recalling (10) and (11) where  $U(t)$  has been replaced by  $U_d(t)$ , we conclude that the actuator state  $v[t]$  is not identically zero on  $t \in [\mu_f, t_f]$ . Recalling Lemmas 2 and 3, and Corollary 1, we thus obtain (67). According to

(61a), (48), we have

$$t_z < t_{z+1} \leq T_f. \quad (68)$$

Recalling  $t_f \leq t_{z+1}$ , this lemma is obtained.  $\blacksquare$

Now, we are in a position to state our main result in the following theorem, i.e., exponential regulation of the plant and actuator states.

**Theorem 2:** For all initial data  $(z[0], w[0])^T \in L^2(0, 1)$ ,  $X(0) \in \mathbb{R}^m$ ,  $\hat{D}(0) \in [\underline{D}, \bar{D}]$ , considering the closed-loop system consisting of the plant (1)–(4), (9)–(12), the controller (61), the triggering mechanism (48), and the least-squares identifier (60), the exponential regulation of the closed-loop system is obtained in the sense that there exist positive constants  $M, \lambda_1$  such that

$$\Omega(t) \leq M e^{-\lambda_1 t}, \quad t \geq 0 \quad (69)$$

where  $\Omega(t)$  is defined in (35).

*Proof.* *Case 1:* Equation (61a) is not executed. Then, the delay-adaptive control law  $U_d$  is (61b) all the time. Replacing the nominal control law  $U$  by  $U_d$  defined by (61b) in (11), through the transformations in Section III, the right boundary condition of the actuator PDE (30) in the target system (17)–(20) and (28)–(30) becomes

$$\hat{u}(0, t) = \xi(t) \quad (70)$$

where

$$\begin{aligned} \xi(t) &= U_d(t; \hat{D}) - U(t; D) \\ &= \int_0^1 (M_1(y; \hat{D}) - M_1(y; D)) z(y, t) dy \\ &\quad + \int_0^1 (M_2(y; \hat{D}) - M_2(y; D)) w(y, t) dy \\ &\quad + \int_0^1 (M_3(y; \hat{D}) - M_3(y; D)) v(y, t) dy \\ &\quad + (M_4(\hat{D}) - M_4(D)) X(t). \end{aligned} \quad (71)$$

Taking the derivative of (36) along the target system states corresponding to the even-based closed-loop system consisting of (17)–(20), (28), (29), and (70), through a similar process in (44), recalling conditions (37)–(40) on  $\delta, r_a, r_c$ , and  $r_d$  [we emphasize that conditions (37)–(40) only depend on the known plant parameters and the known bounds of the unknown parameters in Assumption 3], we obtain

$$\dot{V}(t) \leq -\lambda_1 V(t) + \frac{r_c}{2D} \xi(t)^2, \quad t \geq 0 \quad (72)$$

where  $\lambda_1$  is given in (46). According to (67) and (71), one can establish that

$$\xi(t) \equiv 0, \quad t \in [t_f, \infty). \quad (73)$$

We then have that

$$\dot{V}(t) \leq -\lambda_1 V(t), \quad t \geq t_f. \quad (74)$$

Multiplying both sides of (74) by  $e^{\lambda_1 t}$  and integrating the resulting terms from  $t_f$  to  $t$  lead to the following inequality:

$$V(t) \leq V(t_f) e^{-\lambda_1(t-t_f)}, \quad t \geq t_f$$

which by virtue of (41), is equivalent to

$$\Omega(t) \leq \Upsilon \Omega(t_f) e^{-\lambda_1(t-t_f)}, \quad t \geq t_f \quad (75)$$

where  $\Omega$  is defined in (35) and the positive constant  $\Upsilon$  is given in (47).

Note that the norm estimate (75) is only true for  $t \geq t_f$ . Next, we extend our analysis for  $t \in [0, t_f]$ . With the help of (41), (71), we obtain from (72) that

$$\dot{V}(t) \leq -\lambda_1 V(t) + Q(\hat{D}(0))V(t), \quad t \in [0, t_f] \quad (76)$$

where the positive constant  $Q(\hat{D}(0))$  is

$$\begin{aligned} Q(\hat{D}(0)) &= \max_{y \in [0, 1]} \{(K_1(0, y; \hat{D}(0)) - K_1(0, y; D))^2, \\ &\quad (K_2(0, y; \hat{D}(0)) - K_2(0, y; D))^2, (R(0, y; \hat{D}(0)) - R(0, y; D))^2, \\ &\quad (\eta(0; \hat{D}(0)) - \eta(0; D))^2\} \frac{2r_c}{D\xi_1\xi_3} \end{aligned} \quad (77)$$

which is derived by finding an upper bound for  $\xi(t)^2$  (71) in the form of target states  $\beta, \alpha, \hat{u}$ , and  $X$ , and recalling (41).

Hence, the following holds:

$$\Omega(t) \leq \Upsilon \Omega(0) e^{\lambda_2(\hat{D}(0))t}, \quad t \in [0, t_f] \quad (78)$$

where

$$\lambda_2(\hat{D}(0)) = |Q(\hat{D}(0)) - \lambda_1| > 0$$

and the positive constant  $\Upsilon$  is given in (47). Therefore, it straightforwardly follows that

$$\Omega(t_f) \leq \Upsilon e^{\lambda_2(\hat{D}(0))t_f} \Omega(0). \quad (79)$$

Considering (78), combining (75) and (79) yields

$$\Omega(t) \leq \Upsilon^2 e^{(\lambda_2(\hat{D}(0)) + \lambda_1)t_f} \Omega(0) e^{-\lambda_1 t}, \quad t \geq 0 \quad (80)$$

which is equivalent to (69) with

$$M = \Upsilon^2 e^{(\lambda_2(\hat{D}(0)) + \lambda_1)t_f} \Omega(0).$$

*Case 2:* Equation (61a) is executed. Denoting the time instant  $t_i$  satisfying the condition in (61a) as  $t_z$ , we know from (61) that

$$U_d = \begin{cases} 0, & t \in [0, t_z) \\ r \left( \sin\left(\omega(t - t_i + \frac{\pi}{2\omega})\right) - 1 \right), & t \in [t_z, t_{z+1}) \\ U(t; D), & t \in [t_{z+1}, \infty) \end{cases} \quad (81a)$$

$$U_d = \begin{cases} r \left( \sin\left(\omega(t - t_i + \frac{\pi}{2\omega})\right) - 1 \right), & t \in [t_z, t_{z+1}) \\ U(t; D), & t \in [t_{z+1}, \infty) \end{cases} \quad (81b)$$

$$U_d = \begin{cases} U(t; D), & t \in [t_{z+1}, \infty) \end{cases} \quad (81c)$$

and  $t_f = t_{z+1}$  in Lemma 5, recalling Lemmas 2 and 3 as well as (10) and (11) where  $U(t)$  has been replaced by  $U_d(t)$ . Therefore, following (73)–(75), applying (81c) that implies that  $\xi$  in (71) is identically zero on  $t \geq t_{z+1}$ , we have

$$\Omega(t) \leq \Upsilon \Omega(t_{z+1}) e^{-\lambda_1(t-t_{z+1})}, \quad t \geq t_{z+1}. \quad (82)$$

Following (76)–(79), recalling (81a), we have

$$\Omega(t) \leq \Upsilon \Omega(0) e^{\lambda_3 t}, \quad t \in [0, t_z] \quad (83)$$

for  $t \in [0, t_z]$ , where  $\lambda_3 = Q_1 - \lambda_1$  and the positive constant  $Q_1$  is the one in (77) removing  $K_1(0, y; \hat{D}(0)), K_2(0, y; \hat{D}(0)), R(0, y; \hat{D}(0)), \eta(0; \hat{D}(0))$ . This implies

$$\Omega(t_z) \leq \Upsilon \Omega(0) e^{|\lambda_3| t_z}. \quad (84)$$

Similarly, we obtain from (81b) that

$$\Omega(t) \leq \Upsilon \Omega(t_z) e^{\lambda_4(t-t_z)} + \frac{4r_c r^2 (e^{\lambda_4(t-t_z)} - 1)}{\lambda_4 D \xi_1 \xi_3}, \quad t \in [t_z, t_{z+1}]$$

where  $\lambda_4 = 2Q_1 - \lambda_1$ . Applying (68) and (84), we have

$$\Omega(t_{z+1}) \leq \Upsilon^2 \Omega(0) e^{|\lambda_3| T_f + |\lambda_4| T} + \frac{4r_c r^2 (e^{|\lambda_4| T} - 1)}{|\lambda_4| D \xi_1 \xi_3}. \quad (85)$$

Inserting (85) into (82), one obtains (69) where

$$M = \Upsilon^3 \Omega(0) e^{|\lambda_3| T_f + |\lambda_4| T + \lambda_1 T_f} + \frac{4\Upsilon r_c (e^{|\lambda_4| T} - 1) e^{\lambda_1 T_f}}{|\lambda_4| D \xi_1 \xi_3} r^2. \quad (86)$$

The proof of the theorem is complete.  $\blacksquare$

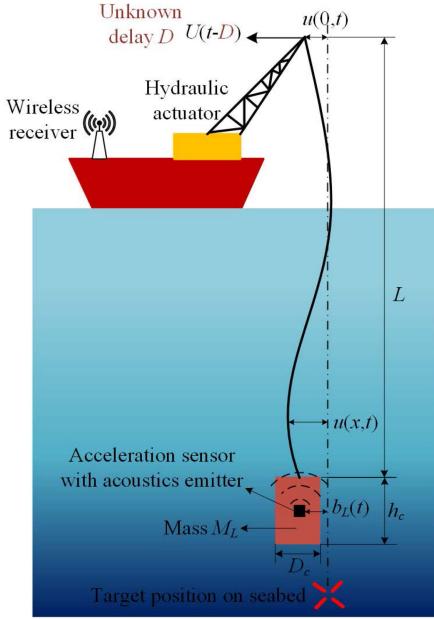


Fig. 2. Deep-sea construction vessel.

## VI. SIMULATION

A DCV is used to place equipment to be installed at the predetermined location on the seafloor, which is shown in Fig. 2. Different from [57] that deals with a known sensor delay that exists in the large-distance transmission of the sensing signal from the seafloor to the vessel on the ocean surface through a set of acoustics devices, we consider all possible delays here (including the transmission of the sensing signal, computation of the control law, and the delay in the hydraulic actuator for the ship-mounted crane and so on) as an unknown delay in the control input channel. By designing a control input at the top of the crane cable, our goal is to reduce the oscillations of the crane cable with the purpose of placing the payload attached at the bottom of the cable in the target area, despite the presence of the unknown delay.

### A. Model

The following dynamic model of cable-payload lateral oscillations in DCV is taken from [57]

$$T_0 u_{\bar{x}}(0, t) = U(t - D) \quad (87)$$

$$\rho u_{tt}(\bar{x}, t) = T_0 u_{\bar{x}\bar{x}}(\bar{x}, t) - d_c u_t(\bar{x}, t) \quad (88)$$

$$u(L, t) = b_L(t) \quad (89)$$

$$M_L \ddot{b}_L(t) = -d_L \dot{b}_L(t) + T_0 u_{\bar{x}}(L, t) \quad (90)$$

$\forall (\bar{x}, t) \in [0, L] \times [0, \infty)$ . The state  $u(\bar{x}, t)$  describes the lateral oscillation displacement along the cable, and  $b_L(t)$  denotes that of the payload. The control input  $U$  is subject to the unknown time delay  $D$  mentioned above. The static tension  $T_0$  is defined as  $T_0 = M_L g - F_{\text{buoyant}}$ , where the buoyancy  $F_{\text{buoyant}}$  is  $F_{\text{buoyant}} = \frac{1}{4}\pi D_c^2 h_c \rho_s g$ . The physical parameters of the DCV are given in Table I.

TABLE I  
PHYSICAL PARAMETERS OF THE DCV

Parameters (units)	values
Cable length $L$ (m)	1500
Cable linear density $\rho$ (kg/m)	7.5
Payload mass $M_L$ (kg)	$3.5 \times 10^5$
Gravitational acceleration $g$ (m/s <sup>2</sup> )	9.8
Cable material damping coefficient $d_c$ (N·s/m)	0.8
Height of payload modeled as a cylinder $h_c$ (m)	7.5
Diameter of payload modeled as a cylinder $D_c$ (m)	5
Damping coefficient at payload $d_L$ (N·s/m)	$1.2 \times 10^5$
Seawater density $\rho_s$ (kg·m <sup>-3</sup> )	1024

Like [57], after applying the Riemann transformations

$$z(\bar{x}, t) = u_t(\bar{x}, t) - \sqrt{\frac{T_0}{\rho}} u_{\bar{x}}(\bar{x}, t) \quad (91)$$

$$w(\bar{x}, t) = u_t(\bar{x}, t) + \sqrt{\frac{T_0}{\rho}} u_{\bar{x}}(\bar{x}, t) \quad (92)$$

introducing a space normalization variable

$$x = \frac{\bar{x}}{L} \in [0, 1] \quad (93)$$

and defining  $X(t) = \dot{b}_L(t)$ , (87)–(90) are rewritten as the considered plant (1)–(5) with the coefficients

$$c_0 = 2\sqrt{\frac{1}{T_0 \rho}}, \quad q_1 = q_2 = \frac{1}{L} \sqrt{\frac{T_0}{\rho}} \quad (94)$$

$$d_1 = d_2 = d_3 = d_4 = \frac{-d_c}{2\rho}, \quad q = -1, \quad p = 1 \quad (95)$$

$$C = 2, \quad A = \frac{-d_L}{M_L} + \frac{\sqrt{T_0 \rho}}{M_L}, \quad B = -\frac{\sqrt{T_0 \rho}}{M_L} \quad (96)$$

which is the simulation model in this section, where it can be checked that the plan parameters in (94)–(96) satisfy Assumptions 1, 2 by recalling Table I.

The initial conditions are defined as

$$z(x, 0) = 8 \sin(5\pi x(1 - x)), \quad w(x, 0) = -8 \cos(5\pi x)$$

thereby,  $X(0) = 1.13$ , recalling (4), which physically corresponds to the initial oscillation velocities of the payload. The unknown delay  $D$  is set as 1, and the known bounds  $\underline{D}$  and  $\overline{D}$  are assumed as 0.01 and 2. We will show the simulation results of the following four cases.

- 1) *Open loop*: The control input is zero.
- 2) *Nonadaptive control*: The nominal delay-compensated control with the unknown delay  $D$  replaced by its estimate 0.25.
- 3) *Delay-adaptive control* with the initial delay estimate  $\hat{D}(0) = 0.25$ , where the design parameter  $K$  in (22) is chosen as  $K = -18$ .
- 4) *Delay-adaptive control* with the initial delay estimate  $\hat{D}(0) = 1.5$ , where the design parameter  $K$  in (22) is chosen as  $K = -13$ .

Other design parameters are chosen as

$$\delta = -0.36, \quad r_a = 1.02, \quad r_c = 1, \quad r_d = 0.02, \quad a = 2,$$

$$T = 3.12, \quad \tilde{N} = 10, \quad T_f = 8, \quad r = 0.5, \quad \omega = 1,$$

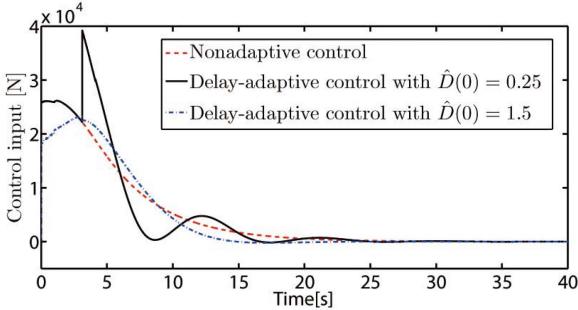


Fig. 3. Delay-adaptive control input  $U_d(t)$  with  $\hat{D}(0) = 0.25$  or  $\hat{D}(0) = 1.5$  and the nonadaptive control input  $U_0(t)$ .

according to (37)–(40), and (49), where  $a, T_f, \tilde{N}$  are free but positive, and where  $r$  and  $\omega$  are free. Actually, like most physical systems, (61a) associated with  $r$  and  $\omega$  has not been activated in the simulation because the control signal is not identically zero for a certain time period from the beginning. The parameter  $\bar{n}$  mentioned in Remark 1 is set as  $\bar{n} = 2$ .

*Remark 3:* In addition to Remark 1 about the implementation of the delay identifier, some more things are worth noting in the simulation. 1) Approximating the integration with respect to the space variable in the identifier as the summation operator will cause a tiny error between the final parameter estimate and the true value in the simulation result, which will be seen in Fig. 4. The smaller space step adopted in the simulation will make the error smaller. 2) The error of approximation in the simulation will also lead to tiny differences between the outputs of the identifier at each updating time even if the effective parameter deification has been achieved. Therefore, we set a small margin to tolerate the approximation error, that is—if the difference between the estimates from the identifier at two adjacent updating times is smaller than 2% of the true value, we consider that this difference is caused by the approximation error in the simulation, and thus keep the estimate value as same as the one at the former updating time.

## B. Simulation Result

The numerical computation is conducted using the finite difference method with the step sizes of  $t$  and  $x$  as 0.001, and 0.02, respectively. The approximate solutions of the kernel PDEs used in the control law, which is defined by (48), (60), (61) where the integral operators are approximated by sums, are also solved by the finite difference method based on the discretization of the triangular domain into a uniformly spaced grid with the interval of 0.02.

The designed delay-adaptive control input and the estimate of the unknown delay are shown in Figs. 3 and 4, respectively, from which we know that the identification of the unknown delay is achieved at the first triggering time, no matter the initial delay estimate is less than ( $\hat{D}(0) = 0.25$ ) or larger than ( $\hat{D}(0) = 1.5$ ) the true value  $D = 1$ . As mentioned in Remark 3, the tiny differences between the delay estimate and its true values come from the error of approximation—that is, approximating the integration with respect to the space variable from 0 to 1 in the identifier as the summation operator for the 51 spatial discrete points with the fixed interval of 0.02. The time evolution of the

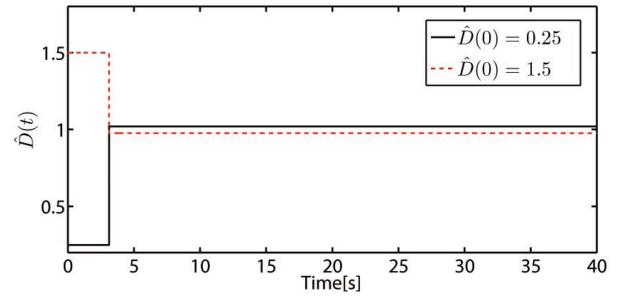


Fig. 4. Estimate of the unknown delay  $D$  under the initial estimate  $\hat{D}(0) = 0.25$  or  $\hat{D}(0) = 1.5$ .

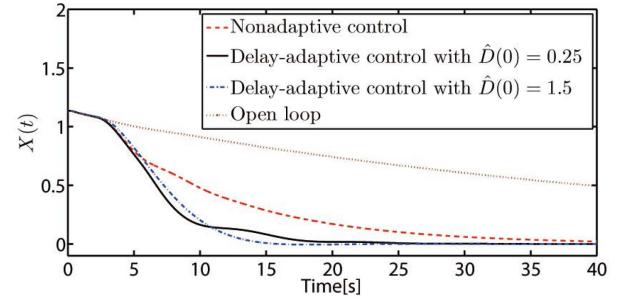


Fig. 5. Evolution of  $X(t)$  under the delay-adaptive control  $U_d(t)$  with  $\hat{D}(0) = 0.25$  or  $\hat{D}(0) = 1.5$  and the nonadaptive control  $U_0(t)$ .

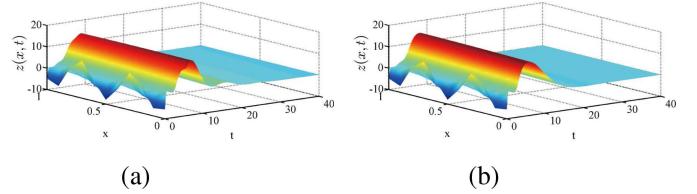


Fig. 6. Evolution of the plant state  $z(x, t)$  under the delay-adaptive control  $U_d(t)$  with  $\hat{D}(0) = 0.25$  or  $\hat{D}(0) = 1.5$ .

ODE state  $X(t)$  is shown in Fig. 5, where the brown dashed line, the red dashed line, the black solid line, and the blue dot-dash line show the results of the four cases mentioned in Section VI-A, respectively. Although both the nonadaptive delay-compensated controller and the delay-adaptive controllers can attenuate the state of the ODE in comparison to the open loop scenario, Fig. 5 further reveals the “delay mismatch” in the nonadaptive control law leads to slower convergence after the time point when the exact delay estimate is obtained and the updated input signal reaches the ODE through the transport PDEs. Even though the simulation model, like many practical models that usually include damping, is not an open-loop unstable plant, the proposed control design still shows improved convergence rates under the proposed delay-adaptive controllers as compared with both the open-loop case and nonadaptive delay-compensated controller. Similarly, it is shown in Figs. 6–8 that the PDE plant states  $z(x, t)$  and  $w(x, t)$ , and the actuator state  $v(x, t)$  all converge to zero when the system is subject to the proposed delay-adaptive control inputs with  $\hat{D}(0) = 0.25$  or  $\hat{D}(0) = 1.5$ .

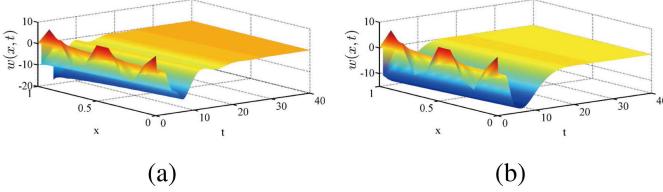


Fig. 7. Evolution of the plant state  $w(x, t)$  under the delay-adaptive control  $U_d(t)$  with  $\hat{D}(0) = 0.25$  or  $\hat{D}(0) = 1.5$ .

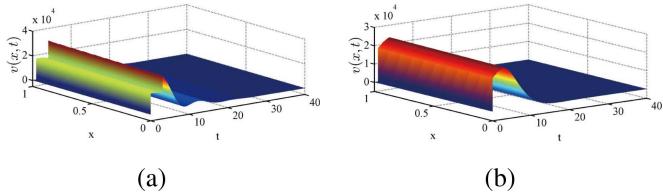


Fig. 8. Evolution of the actuator state  $v(x, t)$  under the delay-adaptive control  $U_d(t)$  with  $\hat{D}(0) = 0.25$  or  $\hat{D}(0) = 1.5$ .

It is easy to obtain the oscillation energy of the cable in DCV  $\frac{\rho}{2}\|u_t(\cdot, t)\|^2 + \frac{T_0}{2}\|u_x(\cdot, t)\|^2 = \frac{\rho}{8}\|w(\cdot, t) + z(\cdot, t)\|^2 + \frac{\rho}{8}\|w(\cdot, t) - z(\cdot, t)\|^2$  by recalling (91)–(93). Therefore, it is known from the results  $z(x, t)$  and  $w(x, t)$  in Figs. 6 and 7 that the oscillation energy of the cable decreases to zero fast under the proposed delay-adaptive controller. One can also observe from Fig. 5 that the regulation performance of the ODE, i.e., the payload, is satisfied.

## VII. CONCLUSION AND FUTURE WORK

In this article, we have proposed a delay-adaptive control scheme for a  $2 \times 2$  hyperbolic PDE-ODE system, where the input delay is arbitrarily large and unknown. The controller consists of a nominal delay-compensated control law, a batch least-squares identifier for the unknown delay, and a triggering mechanism to determine the update times of the identifier. We have proved that the proposed control guarantees the following:

- 1) the avoidance of Zeno phenomenon;
- 2) the identification of the unknown boundary input delay before the prescribed time;
- 3) the exponential regulation of both the plant and the actuator states to zero.

The effectiveness of the proposed design is verified by numerical simulation in the control application of a DCV subject to input delay. This article only deals with the state-feedback adaptive-delay control design for coupled hyperbolic PDEs whose actuator states and plant states are measurable.

In our future work, the output-feedback control design with unmeasurable actuator and plant states, and improvement of the robustness to the sensor measurement error under a short expected identification time, will be considered.

## APPENDIX

### A. Gain Kernels PDEs and Their Associated Boundary Conditions

#### (a) First-step transformation

The backstepping transformation (13) and (14) lead to the following PDE-ODE system of kernel conditions for  $\varphi, \phi, \Psi, \Phi, \gamma$ , and  $\lambda$ . These conditions are derived by mapping the original plant to the first intermediate system

$$\begin{aligned} q_2\varphi_y(x, y) - q_1\varphi_x(x, y) - (d_4 - d_1)\varphi(x, y) \\ - d_2\phi(x, y) = 0 \end{aligned} \quad (97)$$

$$q_1\phi_x(x, y) + q_1\phi_y(x, y) + d_3\varphi(x, y) = 0 \quad (98)$$

$$\begin{aligned} q_2\Psi_x(x, y) - q_1\Psi_y(x, y) + (d_4 - d_1)\Psi(x, y) \\ - d_3\Phi(x, y) = 0 \end{aligned} \quad (99)$$

$$q_2\Phi_x(x, y) + q_2\Phi_y(x, y) - d_2\Psi(x, y) = 0 \quad (100)$$

$$q_1\gamma'(x) + \gamma(x)(A - d_1I_n) + q_1C\phi(x, 0) = 0 \quad (101)$$

$$q_2\lambda'(x) - \lambda(x)(A - d_4I_n) - q_1C\Psi(x, 0) = 0 \quad (102)$$

with the boundary conditions

$$\varphi(x, x) = \frac{d_2}{q_1 + q_2} \quad (103)$$

$$q_2\varphi(x, 0) + q_1p\phi(x, 0) = \gamma(x)B \quad (104)$$

$$\Psi(x, x) = \frac{-d_3}{q_1 + q_2} \quad (105)$$

$$q_2\Phi(x, 0) + q_1p\Psi(x, 0) = \lambda(x)B \quad (106)$$

$$\lambda(0) = K^T \quad (107)$$

$$\gamma(0) = C - pK^T \quad (108)$$

where  $I_n$  is an identity matrix with dimension  $n$ .

Similarly, the boundary conditions of the gain kernels associated with the inverse backstepping transformation (15), (16), namely,  $\bar{\varphi}, \bar{\phi}, \bar{\gamma}, \bar{\Psi}, \bar{\Phi}$ , and  $\bar{\lambda}$  are given by

$$\begin{aligned} q_2\bar{\Psi}_x(x, y) - q_1\bar{\Psi}_y(x, y) + (d_4 - d_1)\bar{\Psi}(x, y) \\ + d_3\bar{\phi}(x, y) = 0 \end{aligned} \quad (109)$$

$$q_1\bar{\phi}_x(x, y) + q_1\bar{\phi}_y(x, y) - d_2\bar{\Psi}(x, y) = 0 \quad (110)$$

$$\begin{aligned} q_2\bar{\varphi}_y(x, y) - q_1\bar{\varphi}_x(x, y) - (d_4 - d_1)\bar{\varphi}(x, y) \\ + d_2\bar{\Phi}(x, y) = 0 \end{aligned} \quad (111)$$

$$q_2\bar{\Phi}_y(x, y) + q_2\bar{\Phi}_x(x, y) + d_3\bar{\varphi}(x, y) = 0 \quad (112)$$

$$q_1\bar{\gamma}'(x) - \bar{\gamma}(x)(A + BK^T + d_1I_n) - d_2\bar{\lambda}(x) = 0 \quad (113)$$

$$q_2\bar{\lambda}'(x) + \bar{\lambda}(x)(A + BK^T + d_4I_n) + d_3\bar{\gamma}(x) = 0 \quad (114)$$

with the boundary conditions

$$\bar{\Psi}(x, x) = -\frac{d_3}{q_1 + q_2} \quad (115)$$

$$q_1p\bar{\phi}(x, 0) + q_2\bar{\varphi}(x, 0) = \bar{\gamma}(x)B \quad (116)$$

$$\bar{\varphi}(x, x) = \frac{d_2}{q_1 + q_2} \quad (117)$$

$$q_2\bar{\Phi}(x, 0) + q_1p\bar{\Psi}(x, 0) = \bar{\lambda}(x)B \quad (118)$$

$$\bar{\lambda}(0) = -K^T \quad (119)$$

$$\bar{\gamma}(0) = pK^T - C. \quad (120)$$

The set of (97)–(108) and (109)–(120) are well-known for coupled linear heterodirectional hyperbolic PDE-ODE systems, and their well-posedness has been proved in [43, Th. 4.1].

(b) *Second-step transformation*

The gain kernels  $K_1$ ,  $K_2$ , and  $\eta$  are defined as follows:

$$dK_{1x}(x, y) + q_1 K_{1y}(x, y) = -d_1 K_1(x, y) \quad (121)$$

$$dK_{2x}(x, y) - q_2 K_{2y}(x, y) = -d_4 K_2(x, y) \quad (122)$$

$$d\eta'(x) A_m^{-1} + \eta(x) = 0 \quad (123)$$

with the boundary conditions

$$K_1(1, y) = \frac{1}{c_0} \bar{\Psi}(1, y) - \frac{1}{c_0} q \bar{\phi}(1, y) \quad (124)$$

$$K_2(1, y) = \frac{1}{c_0} \bar{\Phi}(1, y) - \frac{1}{c_0} q \bar{\varphi}(1, y) \quad (125)$$

$$K_1(x, 1) = \frac{q_2}{q_1} K_2(x, 1) \quad (126)$$

$$q_1 p K_1(x, 0) + q_2 K_2(x, 0) = \eta(x) B \quad (127)$$

$$\eta(1) = -\frac{1}{c_0} q \bar{\gamma}(1) + \frac{1}{c_0} \bar{\lambda}(1). \quad (128)$$

The proof of well-posedness of (121)–(128) is given in [7, Lemma 2]. To derive conditions (121)–(128), one needs to consider (21) and (23). Hence, (24) holds straightforwardly under the conditions (124), (125), and (128). Taking the time and spatial derivatives of (23), inserting the results into (25), recalling (17)–(20) and (24), one obtains

$$\begin{aligned} & u_t(x, t) + du_x(x, t) - q_2 K_2(x, 1) c_0 u(1, t) \\ &= v_t(x, t) + \int_0^1 K_1(x, y) \alpha_t(y, t) dy + \int_0^1 K_2(x, y) \beta_t(y, t) dy \\ &+ dv_x(x, t) + d \int_0^1 K_{1x}(x, y) \alpha(y, t) dy \\ &+ d \int_0^1 K_{2x}(x, y) \beta(y, t) dy \\ &+ \eta(x) \dot{X}(t) + d\eta'(x) X(t) - q_2 K_2(x, 1) c_0 u(1, t) \\ &= v_t(x, t) - q_1 \int_0^1 K_1(x, y) \alpha_x(y, t) dy \\ &+ d_1 \int_0^1 K_1(x, y) \alpha(y, t) dy \\ &+ q_2 \int_0^1 K_2(x, y) \beta_x(y, t) dy + d_4 \int_0^1 K_2(x, y) \beta(y, t) dy \\ &+ dv_x(x, t) + d \int_0^1 K_{1x}(x, y) \alpha(y, t) dy \\ &+ d \int_0^1 K_{2x}(x, y) \beta(y, t) dy + \eta(x) (A_m X(t) + B \beta(0, t)) \\ &+ d\eta'(x) X(t) - q_2 K_2(x, 1) c_0 u(1, t) \\ &= (q_2 K_2(x, 1) q - q_1 K_1(x, 1)) \alpha(1, t) \\ &+ \int_0^x (q_1 K_{1y}(x, y) + d_1 K_1(x, y) + d K_{1x}(x, y)) \alpha(y, t) dy \\ &- (q_2 K_2(x, 0) + q_1 K_1(x, 0) p - \eta(x) B) \beta(0, t) \end{aligned}$$

$$\begin{aligned} & + \int_0^x (d_4 K_2(x, y) - q_2 K_{2y}(x, y) + d K_{2x}(x, y)) \beta(y, t) dy \\ & + (\eta(x) A_m + d\eta'(x)) X(t) = 0. \end{aligned} \quad (129)$$

The necessary and sufficient conditions for (129) to hold are given as (121)–(123), (126), and (127).

(c) *Third-step transformation*

The derivation of the gain kernels PDE  $R$  and  $R^I$  is performed as follows. Substituting the time and spatial derivatives of (27) into (25) and recalling (28)–(30), we have

$$\begin{aligned} & u_t(x, t) + du_x(x, t) - q_2 K_2(x, 1) c_0 u(1, t) \\ &= \hat{u}_t(x, t) + \int_x^1 R(x, y) \hat{u}_t(y, t) dy + d \hat{u}_x(x, t) \\ &+ d \int_x^1 R_x(x, y) \hat{u}(y, t) dy - dR(x, x) \hat{u}(x, t) \\ &- q_2 K_2(x, 1) c_0 \hat{u}(1, t) \\ &= -d \int_x^1 R(x, y) \hat{u}_x(y, t) dy + d \int_x^1 R_x(x, y) \hat{u}(y, t) dy \\ &- dR(x, x) \hat{u}(x, t) - q_2 K_2(x, 1) c_0 \hat{u}(1, t) \\ &= -(dR(x, 1) + q_2 K_2(x, 1) c_0) \hat{u}(1, t) \\ &+ d \int_x^1 (R_x(x, y) + R_y(x, y)) \hat{u}(y, t) dy = 0. \end{aligned} \quad (130)$$

For (130) to hold, the following equality must be satisfied:

$$R_x(x, y) + R_y(x, y) = 0 \quad (131)$$

$$dR(x, 1) = -q_2 c_0 K_2(x, 1) \quad (132)$$

which obviously admits a unique solution

$$R(x, y) = -\frac{q_2 c_0}{d} K_2(x - y + 1, 1).$$

Similarly, substituting the time and spatial derivatives of (32) into (29) and recalling (25), we have

$$\begin{aligned} & \hat{u}_t(x, t) + d \hat{u}_x(x, t) \\ &= u_t(x, t) + \int_x^1 P(x, y) u_t(y, t) dy + du_x(x, t) \\ &+ d \int_x^1 P_x(x, y) u(y, t) dy - dP(x, x) u(x, t) \\ &= q_2 K_2(x, 1) c_0 u(1, t) - d \int_x^1 P(x, y) u_x(y, t) dy \\ &+ \int_x^1 P(x, y) q_2 K_2(y, 1) c_0 dy u(1, t) \\ &+ d \int_x^1 P_x(x, y) u(y, t) dy - dP(x, x) u(x, t) \\ &= \left( q_2 K_2(x, 1) c_0 - dP(x, 1) \right. \\ &\quad \left. + \int_x^1 P(x, y) q_2 K_2(y, 1) c_0 dy \right) u(1, t) \\ &+ d \int_x^1 (P_y(x, y) + P_x(x, y)) u(y, t) dy = 0. \end{aligned}$$

The equation above suggests that the kernel function  $P$  in the inverse transformation (32) satisfies the following PDE with the

corresponding boundary value:

$$P_y(x, y) - P_x(x, y) = 0 \quad (133)$$

$$P(x, 1) = \frac{q_2}{d} K_2(x, 1) c_0 + \frac{1}{d} \int_x^1 P(x, y) q_2 K_2(y, 1) c_0 dy \quad (134)$$

whose well-posedness can be obtained by the method of characteristics.

### B. Expressions of the Controller Gain Functions $M_1, M_2, M_3, M_4$

The functions  $M_1, M_2, M_3$ , and  $M_4$  are given as follows:

$$\begin{aligned} M_1(y) = & \int_0^1 R(0, s) K_1(s, y) ds - K_1(0, y) \\ & + \int_0^1 R(0, s_1) \int_{s_1}^1 P(s_1, s) K_1(s, y) ds ds_1 \\ & - \int_y^1 \left[ \int_0^1 R(0, y) K_1(y, s) dy - K_1(0, s) \right. \\ & \left. + \int_0^1 R(0, s_1) \int_{s_1}^1 P(s_1, y) K_1(y, s) dy ds_1 \right] \phi(s, y) ds \\ & - \int_y^1 \left[ \int_0^1 R(0, y) K_2(y, s) dy - K_2(0, s) \right. \\ & \left. + \int_0^1 R(0, s_1) \int_{s_1}^1 P(s_1, y) K_2(y, s) dy ds_1 \right] \Psi(s, y) ds \end{aligned}$$

$$\begin{aligned} M_2(y) = & - \int_y^1 \left[ \int_0^1 R(0, y) K_1(y, s) dy - K_1(0, s) \right. \\ & \left. + \int_0^1 R(0, s_1) \int_{s_1}^1 P(s_1, y) K_1(y, s) dy ds_1 \right] \varphi(s, y) ds \\ & + \int_0^1 R(0, s) K_2(s, y) ds - K_2(0, y) \\ & + \int_0^1 R(0, s_1) \int_{s_1}^1 P(s_1, s) K_2(s, y) ds ds_1 \\ & - \int_y^1 \left[ \int_0^1 R(0, y) K_2(y, s) dy - K_2(0, s) \right. \\ & \left. + \int_0^1 R(0, s_1) \int_{s_1}^1 P(s_1, y) K_2(y, s) dy ds_1 \right] \Phi(s, y) ds \end{aligned}$$

$$M_3(y) = R(0, y) + \int_0^y R(0, s) P(s, y) ds$$

$$\begin{aligned} M_4 = & \int_0^1 R(0, y) \eta(y) dy - \eta(0) \\ & + \int_0^1 R(0, y) \int_y^1 P(y, s) \eta(s) ds dy \\ & - \int_0^1 \left[ \int_0^1 R(0, s) K_2(s, y) ds - K_2(0, y) \right. \\ & \left. + \int_0^1 R(0, s_1) \int_{s_1}^1 P(s_1, s) K_2(s, y) ds ds_1 \right] \lambda(y) dy \end{aligned}$$

$$\begin{aligned} & - \int_0^1 \left[ \int_0^1 R(0, s) K_1(s, y) ds - K_1(0, y) \right. \\ & \left. + \int_0^1 R(0, s_1) \int_{s_1}^1 P(s_1, s) K_1(s, y) ds ds_1 \right] \gamma(y) dy \end{aligned}$$

where  $K_1, K_2, \eta, R$ , and  $P$  are parameterized by the unknown delay  $D = \frac{1}{d}$  according to the conditions defined in Appendix A.

### C. Proof of Proposition 1

*Case 1:* With the delay-adaptive control input  $U_d$  (61b), applying the following transformations:

$$\alpha(x, t) = e^{\frac{d_1}{q_1}x} \bar{\alpha}(x, t) \quad (135)$$

$$\beta(x, t) = e^{-\frac{d_4}{q_2}x} \bar{\beta}(x, t) \quad (136)$$

the target system, i.e., equivalently closed-loop system, is written as

$$\dot{X}(t) = A_m X(t) + B \bar{\beta}(0, t) \quad (137)$$

$$\bar{\alpha}(0, t) = -p \bar{\beta}(0, t) \quad (138)$$

$$\bar{\alpha}_t(x, t) = -q_1 \bar{\alpha}_x(x, t) \quad (139)$$

$$\bar{\beta}_t(x, t) = q_2 \bar{\beta}_x(x, t) \quad (140)$$

$$\bar{\beta}(1, t) = c_0 e^{\frac{d_4}{q_2}t} \hat{u}(1, t) + q e^{\frac{d_1}{q_1} + \frac{d_4}{q_2}t} \bar{\alpha}(1, t) \quad (141)$$

$$\hat{u}_t(x, t) = -d \hat{u}_x(x, t) \quad (142)$$

$$\begin{aligned} \hat{u}(0, t) = & - \int_0^1 \mathcal{K}_{1i}(y) \bar{\alpha}(y, t) dy - \int_0^1 \mathcal{K}_{2i}(y) \bar{\beta}(y, t) dy \\ & + \int_0^1 \mathcal{R}_i(y) \hat{u}(y, t) dy - \eta_i X(t) \end{aligned} \quad (143)$$

for  $t \in [t_i, t_{i+1}]$  where  $\mathcal{K}_{1i}(y) = (K_1(0, y; \hat{D}(t_i)) - K_1(0, y; D)) e^{\frac{d_1}{q_1}y}$ ,  $\mathcal{K}_{2i}(y) = (K_2(0, y; \hat{D}(t_i)) - K_2(0, y; D)) e^{-\frac{d_4}{q_2}y}$ ,  $\mathcal{R}_i(y) = R(0, y; \hat{D}(t_i)) - R(0, y; D)$ , and  $\eta_i = \eta(0; \hat{D}(t_i)) - \eta(0; D)$  considering (31).

Next, we prove the well-posedness of (137)–(143) by the method of characteristics following [16]. Considering a constant  $0 < \bar{T}_i < \min\{t_{i+1} - t_i, \frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{d}\}$ , by the method of characteristics, for  $\varsigma \in [0, \bar{T}_i]$  we get

$$\bar{\beta}(x, t_i + \varsigma) = \begin{cases} \bar{\beta}(x + q_2 \varsigma, t_i), & x < 1 - q_2 \varsigma \\ c_0 e^{\frac{d_4}{q_2} \hat{u}(1 - d(\varsigma - \frac{1-x}{q_2}), t_i)} \\ + q e^{\frac{d_1}{q_1} + \frac{d_4}{q_2} \bar{\alpha}(1 - q_1(\varsigma - \frac{1-x}{q_2}), t_i)}, & x \geq 1 - q_2 \varsigma \end{cases} \quad (144)$$

$$\bar{\alpha}(x, t_i + \varsigma) = \begin{cases} \bar{\alpha}(x - q_1 \varsigma, t_i), & x > q_1 \varsigma \\ -p \bar{\beta}(q_2(\varsigma - \frac{x}{q_1}), t_i), & x \leq q_1 \varsigma. \end{cases} \quad (145)$$

Integrating (137) and recalling (144), we have

$$X(t_i + \varsigma) = A_m \int_{t_i}^{t_i + \varsigma} X(\tau) d\tau + \int_0^\varsigma B \bar{\beta}(q_2 \tau, t_i) d\tau + X(t_i) \quad (146)$$

for  $\varsigma \in [0, \bar{T}_i]$ . According to (142) and (143), we also obtain by the method of characteristics that

$$\hat{u}(x, t_i + \varsigma) = \hat{u}(x - d\varsigma, t_i), \quad x > d\varsigma \quad (147)$$

and the solution  $\hat{u}(x, t_i + \varsigma)$  for  $x \leq d\varsigma$  is given by

$$\begin{aligned} \hat{u}(x, t_i + \varsigma) &= \hat{u}(0, t_i + t^*) \\ &= - \int_0^1 \mathcal{K}_{1i}(y) \bar{\alpha}(y, t_i + t^*) dy - \int_0^1 \mathcal{K}_{2i}(y) \bar{\beta}(y, t_i + t^*) dy \\ &\quad + \int_0^1 \mathcal{R}_i(y) \hat{u}(y, t_i + t^*) dy - \eta_i X(t_i + t^*) \\ &= s(t_i + t^*) + \int_0^{dt^*} \mathcal{R}_i(y) \hat{u} \left( 0, t_i + t^* - \frac{y}{d} \right) dy, \quad x \leq d\varsigma \end{aligned} \quad (148)$$

where  $t^* = \varsigma - \frac{x}{d} \in [0, \varsigma]$ , and where

$$\begin{aligned} s(t_i + t^*) &= - \int_0^1 \mathcal{K}_{1i}(y) \bar{\alpha}(y, t_i + t^*) dy \\ &\quad - \int_0^1 \mathcal{K}_{2i}(y) \bar{\beta}(y, t_i + t^*) dy \\ &\quad - \eta_i X(t_i + t^*) + \int_{dt^*}^1 \mathcal{R}_i(y) \hat{u}(y - dt^*, t_i) dy. \end{aligned} \quad (149)$$

Applying (144)–(146), one obtains

$$\begin{aligned} s(t_i + t^*) &= \int_0^{q_1 t^*} \mathcal{K}_{1i}(y) p \bar{\beta} \left( q_2 \left( t^* - \frac{y}{q_1} \right), t_i \right) dy \\ &\quad - \int_{q_1 t^*}^1 \mathcal{K}_{1i}(y) \bar{\alpha}(y - q_1 t^*, t_i) dy \\ &\quad - \int_0^{1-q_2 t^*} \mathcal{K}_{2i}(y) \bar{\beta}(y + q_2 t^*, t_i) dy \\ &\quad - \int_{1-q_2 t^*}^1 \mathcal{K}_{2i}(y) \left( c_0 e^{\frac{d_4}{q_2} y} \hat{u} \left( 1 - d \left( t^* - \frac{1-y}{q_2} \right), t_i \right) \right. \\ &\quad \left. + q e^{\frac{d_1}{q_1} + \frac{d_4}{q_2}} \bar{\alpha} \left( 1 - q_1 \left( t^* - \frac{1-y}{q_2} \right), t_i \right) \right) dy \\ &\quad - \eta_i \left( A_m \int_{t_i}^{t_i + t^*} X(\tau) d\tau \right. \\ &\quad \left. + \int_0^{t^*} B \bar{\beta}(q_2 \tau, t_i) d\tau + X(t_i) \right) \\ &\quad + \int_{dt^*}^1 \mathcal{R}_i(y) \hat{u}(y - dt^*, t_i) dy. \end{aligned} \quad (150)$$

Recalling  $(\bar{\alpha}[t_i], \bar{\beta}[t_i], \hat{u}[t_i])^T \in L^2((0, 1); \mathbb{R}^3)$ ,  $X(t_i) \in \mathbb{R}^m$  ensured by the initial condition  $(z[t_i], w[t_i], v[t_i])^T \in L^2((0, 1); \mathbb{R}^3)$ ,  $X(t_i) \in \mathbb{R}^m$  and the transformations (13), (14), (23), (32), (135), and (136), it is obtained from (150) that  $s(t_i + t^*)$  is well-defined.

Defining  $\varrho = t^* - \frac{y}{d}$ , we obtain from (148) that

$$\hat{u}(0, t_i + t^*) = s(t_i + t^*) + d \int_0^{t^*} \mathcal{R}_i(d(t^* - \varrho)) \hat{u}(0, t_i + \varrho) d\varrho \quad (151)$$

for  $t^* \in [0, \varsigma]$ . Since  $s(t_i + t^*)$  and  $\mathcal{R}_i(d(t^* - \varrho))$  are well defined for  $t^* \in [0, \varsigma]$ , where  $\varsigma \in [0, \bar{T}_i]$ , and in addition  $\mathcal{R}_i(d(t^* - \varrho))$  is also continuous in the interval, then (151) is a linear Volterra integral equation with a unique solution (see [29, Th. 5]). Recalling (144)–(148) and  $(\bar{\alpha}[t_i], \bar{\beta}[t_i], \hat{u}[t_i])^T \in L^2((0, 1); \mathbb{R}^3)$ ,  $X(t_i) \in \mathbb{R}^m$ , we obtain

the well-posedness result in the sense of  $((\bar{\alpha}, \bar{\beta}, \hat{u})^T, X) \in C^0([t_i, \bar{T}_i]; L^2(0, 1); \mathbb{R}^3) \times C^0([t_i, \bar{T}_i]; \mathbb{R}^m)$ . Then, starting from  $(\bar{\alpha}[\bar{T}_i], \bar{\beta}[\bar{T}_i], \hat{u}[\bar{T}_i])^T \in L^2((0, 1); \mathbb{R}^3)$ ,  $X(\bar{T}_i) \in \mathbb{R}^m$ , repeating the above process step by step, we obtain  $((\bar{\alpha}, \bar{\beta}, \hat{u})^T, X) \in C^0([t_i, t_{i+1}]; L^2(0, 1); \mathbb{R}^3) \times C^0([t_i, t_{i+1}]; \mathbb{R}^m)$ . Recalling the transformations (15), (16), (23), (27), (135), and (136), Proposition 1 is thus obtained.

*Case 2:* With the delay-adaptive control input  $U_d$  (61a), the only difference from Case 1 is that the left boundary condition of  $\hat{u}$  becomes

$$\begin{aligned} \hat{u}(0, t) &= - \int_0^1 \bar{K}_{1z}(y) \bar{\alpha}(y, t) dy - \int_0^1 \bar{K}_{2z}(y) \bar{\beta}(y, t) dy \\ &\quad + \int_0^1 \bar{R}_z(y) \hat{u}(y, t) dy - \bar{\eta}_z X(t) \\ &\quad + r \left( \sin \left( \omega \left( t - t_z + \frac{\pi}{2\omega} \right) \right) - 1 \right) \end{aligned}$$

for  $t \in [t_z, t_{z+1}]$ , where  $\bar{K}_{1z}(y) = -K_1(0, y; D) e^{\frac{d_1}{q_1} y}$ ,  $\bar{K}_{2z}(y) = -K_2(0, y; D) e^{-\frac{d_4}{q_2} y}$ ,  $\bar{R}_z(y) = -R(0, y; D)$ , and  $\bar{\eta}_z = -\eta(0; D)$ . This difference introduces an additional term  $r(\sin(\omega(t^* + \frac{\pi}{2\omega})) - 1)$ , which is well-defined, into  $s(t_z + t^*)$  in (150) (replacing  $\mathcal{K}, \mathcal{R}, \eta$  by  $\bar{K}, \bar{R}, \bar{\eta}$ ). Therefore,  $s(t_z + t^*)$  is still well-defined, and thus well-posedness result in Case 1 still holds.

#### D. Norms equivalence between the original and the target systems' states

From (13)–(16), (27), and (32), we get

$$\|\alpha(\cdot, t)\|^2 \leq \eta_1 \left( \|z(\cdot, t)\|^2 + \|w(\cdot, t)\|^2 + |X(t)|^2 \right) \quad (152)$$

$$\|\beta(\cdot, t)\|^2 \leq \eta_2 \left( \|z(\cdot, t)\|^2 + \|w(\cdot, t)\|^2 + |X(t)|^2 \right) \quad (153)$$

$$\|z(\cdot, t)\|^2 \leq \eta_3 \left( \|\alpha(\cdot, t)\|^2 + \|\beta(\cdot, t)\|^2 + |X(t)|^2 \right) \quad (154)$$

$$\|w(\cdot, t)\|^2 \leq \eta_4 \left( \|\alpha(\cdot, t)\|^2 + \|\beta(\cdot, t)\|^2 + |X(t)|^2 \right) \quad (155)$$

$$\|u(x, t)\|^2 \leq \eta_5 \|\hat{u}(x, t)\|^2 \quad (156)$$

$$\|\hat{u}(x, t)\|^2 \leq \eta_6 \|u(x, t)\|^2 \quad (157)$$

where

$$\begin{aligned} \eta_1 &= 4 \left( 1 + \int_0^1 \int_0^x \phi(x, y)^2 dy dx + \int_0^1 \int_0^x \varphi(x, y)^2 dy dx \right. \\ &\quad \left. + \int_0^1 \gamma(x)^2 dx \right) \end{aligned}$$

$$\begin{aligned} \eta_2 &= 4 \left( 1 + \int_0^1 \int_0^x \Psi(x, y)^2 dy dx + \int_0^1 \int_0^x \Phi(x, y)^2 dy dx \right. \\ &\quad \left. + \int_0^1 \lambda(x)^2 dx \right) \end{aligned}$$

$$\begin{aligned} \eta_3 &= 4 \left( 1 + \int_0^1 \int_0^x \bar{\phi}(x, y)^2 dy dx + \int_0^1 \int_0^x \bar{\varphi}(x, y)^2 dy dx \right. \\ &\quad \left. + \int_0^1 \bar{\gamma}(x)^2 dx \right) \end{aligned}$$

$$\begin{aligned}\eta_4 &= 4 \left( 1 + \int_0^1 \int_0^x \bar{\Psi}(x, y)^2 dy dx + \int_0^1 \int_0^x \bar{\Phi}(x, y)^2 dy dx \right. \\ &\quad \left. + \int_0^1 \bar{\lambda}(x)^2 dx \right) \\ \eta_5 &= 2 \left( 1 + \int_0^1 \int_x^1 R(x, y)^2 dy dx \right) \\ \eta_6 &= 2 \left( 1 + \int_0^1 \int_x^1 P(x, y)^2 dy dx \right).\end{aligned}$$

Recalling (23), together with (152), (153), (156), and (157), yields

$$\begin{aligned}\|v(\cdot, t)\|^2 &\leq 4 \left( \eta_5 + \int_0^1 \int_0^1 K_1(x, y)^2 dy dx \right. \\ &\quad \left. + \int_0^1 \int_0^1 K_2(x, y)^2 dy dx + \int_0^1 \eta(x)^2 dx \right) \\ &\quad \times (\|\hat{u}(\cdot, t)\|^2 + \|\alpha(\cdot, t)\|^2 + \|\beta(\cdot, t)\|^2 + |X(t)|^2)\end{aligned}\quad (158)$$

$$\begin{aligned}\|\hat{u}(\cdot, t)\|^2 &\leq 4\eta_6(\eta_1 + \eta_2 + 1) \left( 1 + \int_0^1 \int_0^1 K_1(x, y)^2 dy dx \right. \\ &\quad \left. + \int_0^1 \int_0^1 K_2(x, y)^2 dy dx + \int_0^1 \eta(x)^2 dx \right) \\ &\quad \times (\|v(\cdot, t)\|^2 + \|z(\cdot, t)\|^2 + \|w(\cdot, t)\|^2 + |X(t)|^2).\end{aligned}\quad (159)$$

Defining

$$\bar{\Omega}(t) = \|\alpha[t]\|^2 + \|\beta[t]\|^2 + \|\hat{u}[t]\|^2 + |X(t)|^2 \quad (160)$$

one obtains

$$\xi_1 \Omega(t) \leq \bar{\Omega}(t) \leq \xi_2 \Omega(t) \quad (161)$$

where

$$\begin{aligned}\xi_1 &= 1 / \left( 1 + \eta_3 + \eta_4 + 4\eta_5 + 4 \int_0^1 \int_0^1 K_1(x, y)^2 dy dx \right. \\ &\quad \left. + 4 \int_0^1 \int_0^1 K_2(x, y)^2 dy dx + 4 \int_0^1 \eta(x)^2 dx \right) \quad (162)\end{aligned}$$

$$\begin{aligned}\xi_2 &= 1 + \eta_1 + \eta_2 \\ &\quad + 4\eta_6(\eta_1 + \eta_2 + 1) \left( 1 + \int_0^1 \int_0^1 K_1(x, y)^2 dy dx \right. \\ &\quad \left. + \int_0^1 \int_0^1 K_2(x, y)^2 dy dx + \int_0^1 \eta(x)^2 dx \right).\end{aligned}\quad (163)$$

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